

# DOUBLE LOOP QUANTUM ENVELOPING ALGEBRAS

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**ABSTRACT.** In this paper we describe certain homological properties and representations of a two-parameter quantum enveloping algebra  $U_{g,h}$  of  $\mathfrak{sl}(2)$ , where  $g, h$  are group-like elements.

## 1. INTRODUCTION

It is well-known that there is a bijective map  $L \rightarrow P_L$  from the set of all oriented links  $L$  in  $\mathbb{R}^3$  to the ring  $\mathbb{Z}[g^{\pm 1}, h^{\pm 1}]$  of two-variable Laurent polynomials.  $P_L$  is called the Jones-Conway polynomial of the link  $L$ . The Jones-Conway polynomial is an isotopy invariant of oriented links satisfying what knot theorists call “skein relations” (see [11]). Suppose  $\mathbb{K}$  is a field with characteristic zero and  $q$  is a nonzero element in  $\mathbb{K}$  satisfying  $q^2 \neq 1$ . Let  $U_q(\mathfrak{sl}(2))$  be the usual quantum enveloping algebra of the Lie algebra  $\mathfrak{sl}(2)$  with generators  $E, F, K^{\pm 1}$ . Then the vector space

$$U_{g,h} := \mathbb{K}[g^{\pm 1}, h^{\pm 1}] \otimes_{\mathbb{K}} U_q(\mathfrak{sl}(2))$$

has been endowed with a Hopf algebra structure in [13].

We abuse notation and write  $g^{\pm 1}, h^{\pm 1}, E, F, K^{\pm 1}$  for  $g^{\pm 1} \otimes 1, h^{\pm 1} \otimes 1, 1 \otimes E, 1 \otimes F, 1 \otimes K^{\pm 1}$  respectively. In addition,  $g^{\pm 1}, h^{\pm 1}, K^{\pm 1}$  are abbreviated to  $g, h, K$  respectively. Then  $U_{g,h}$  is an algebra over  $\mathbb{K}$  generated by  $g, g^{-1}, h, h^{-1}, E, F, K, K^{-1}$ . These generators satisfy the following relations.

$$(1.1) \quad K^{-1}K = KK^{-1} = 1, \quad g^{-1}g = gg^{-1} = 1, \quad h^{-1}h = hh^{-1} = 1,$$

$$(1.2) \quad KEK^{-1} = q^2E, \quad gh = hg, \quad gK = Kg, \quad gE = Eg, \quad hE = Eh,$$

$$(1.3) \quad KFK^{-1} = q^{-2}F, \quad hK = Kh, \quad hF = Fh, \quad gF = Fg,$$

$$(1.4) \quad EF - FE = \frac{K - K^{-1}g^2}{q - q^{-1}}.$$

The other operations of the Hopf algebra  $U_{g,h}$  are defined as follows:

$$(1.5) \quad \Delta(E) = h^{-1} \otimes E + E \otimes hK,$$

$$(1.6) \quad \Delta(F) = K^{-1}hg^2 \otimes F + F \otimes h^{-1},$$

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$$(1.7) \quad \Delta(K) = K \otimes K, \quad \Delta(K^{-1}) = K^{-1} \otimes K^{-1},$$

$$(1.8) \quad \Delta(a) = a \otimes a, \quad a \in G,$$

where  $G = \{g^m h^n | m, n \in \mathbb{Z}\}$ ,

$$(1.9) \quad \varepsilon(K) = \varepsilon(K^{-1}) = \varepsilon(a) = 1, \quad a \in G,$$

$$(1.10) \quad \varepsilon(E) = \varepsilon(F) = 0,$$

and

$$(1.11) \quad S(E) = -EK^{-1}, \quad S(F) = -KFg^{-2},$$

$$(1.12) \quad S(a) = a^{-1}, \quad a \in G, \quad S(K) = K^{-1}, \quad S(K^{-1}) = K.$$

The Hopf algebra  $U_{g,h}$  is a special case of the Hopf algebras defined in [14]. It is isomorphic to the tensor product of  $U_q(\mathfrak{sl}(2))$  and  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}]$  as algebras. However, the coproduct of  $U_{g,h}$  is not the usual coproduct of the tensor product of two coalgebras. Neither is the antipode.

Homological methods have been used to study Hopf algebras by many authors (see [2], [15] and their references). However, there are few examples of Hopf algebras satisfying a given set of homological properties. In this paper, we describe certain homological properties of the Hopf algebra  $U_{g,h}$  and consequently give an example satisfying some homological properties. Moreover, we study the representation theory of the algebra  $U_{g,h}$ . Similar to [7] and [8], we can define some version of the Bernstein-Gelfand-Gelfand (abbreviated as BGG) category  $\mathcal{O}$ . Furthermore, we decompose the BGG category  $\mathcal{O}$  into a direct sum of subcategories, which are equivalent to categories of finitely generated modules over some finite-dimensional algebras.

Let us outline the structure of this paper. In Section 2, we study the homological properties of  $U_{g,h}$ . We prove that  $U_{g,h}$  is Auslander-regular and Cohen-Macaulay, and the global dimension and Gelfand-Kirillov dimension of  $U_{g,h}$  are equal. We also prove that the center of  $U_{g,h}$  is equal to  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}, C]$ , where  $C$  is the Casimir element of  $U_{g,h}$ . To study the category  $\mathcal{O}$  in Section 4, we prove that  $U_{g,h}$  has an anti-involution that acts as the identity on all of  $\mathbb{K}[K^{\pm 1}, g^{\pm 1}, h^{\pm 1}]$ .

Since there is a finite-dimensional non-semisimple module over the algebra  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}]$ , there is a finite-dimensional non-semisimple module over  $U_{g,h}$ . In Section 3, we compute the extension group  $\text{Ext}^1(M, M')$  in the case that the nonzero  $q$  is not a root of unity, where  $M, M'$  are finite-dimensional simple modules over  $U_{g,h}$ . We prove that the tensor functor  $V \otimes -$  determines an isomorphism from  $\text{Ext}^1(\mathbb{K}_{\alpha', \beta'}, \mathbb{K}_{\alpha, \beta})$  to  $\text{Ext}^1(V \otimes \mathbb{K}_{\alpha', \beta'}, V \otimes \mathbb{K}_{\alpha, \beta})$  for any finite-dimensional simple  $U_q(\mathfrak{sl}(2))$ -module  $V$ . We also obtain a decomposition theory about the tensor product of two simple  $U_{g,h}$ -modules. From this, we obtain a Hopf subalgebra of the finite dual Hopf algebra  $U_{g,h}^\circ$  of  $U_{g,h}$ , which is generated by coordinate functions of finite-dimensional simple modules of  $U_{g,h}$ .

In Section 4, we briefly discuss the Verma modules of  $U_{g,h}$ . The BGG subcategory  $\mathcal{O}$  of the category of representations of  $U_{g,h}$  is introduced and studied. The main results in [8] also hold in the category  $\mathcal{O}$  over the algebra  $U_{g,h}$ .

Throughout this paper  $\mathbb{K}$  is a fixed algebraically closed field with characteristic zero;  $\mathbb{N}$  is the set of natural numbers;  $\mathbb{Z}$  is the set of all integers.  $*^{+1}$  is usually abbreviated to  $*$ . All modules over a ring  $R$  are left  $R$ -modules.

It is worth mentioning that some results of this article are also true if  $\mathbb{K}$  is not an algebraically closed field. We always assume that  $\mathbb{K}$  is an algebraically closed field for simplicity throughout this paper.

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## 2. SOME PROPERTIES OF $U_{g,h}$

In this section, we firstly prove that  $U_{g,h}$  is a Noetherian domain with a PBW basis. Then we compute the global dimension and Gelfand-Kirillov dimension of  $U_{g,h}$ . Moreover, we show that  $U_{g,h}$  is Auslander regular, Auslander Gorenstein, Cohen-Macaulay and Tdeg-stable. For the undefined terms in this section, we refer the reader to [2] and [3].

**Theorem 2.1** (PBW Theorem). *The algebra  $U_{g,h}$  is a Noetherian domain. Moreover, it has a PBW basis  $\{F^l K^m g^n h^s E^t | l, t \in \mathbb{Z}_{\geq 0}; m, n, s \in \mathbb{Z}\}$ .*

*Proof.* Let  $R = \mathbb{K}[K^{\pm 1}, g^{\pm 1}, h^{\pm 1}]$ . Since  $R$  is a homomorphic image of the polynomial ring  $\mathbb{K}[x_1, x_2, \dots, x_5, x_6]$  ( $\varphi(x_1) = K$ ,  $\varphi(x_2) = K^{-1}$ ,  $\varphi(x_3) = g$ ,  $\varphi(x_4) = g^{-1}$ ,  $\varphi(x_5) = h$ ,  $\varphi(x_6) = h^{-1}$ ),  $R$  is a Noetherian ring with a basis  $\{K^m g^n h^s | m, n, s \in \mathbb{Z}\}$ . It is easy to prove that  $R$  is a domain.

Define  $\sigma(K^c g^a h^b) = q^{2c} K^c g^a h^b$ ,  $\forall a, b, c \in \mathbb{Z}$ ,  $\delta(R) \equiv 0$ , and extend  $\sigma$  by additivity and multiplicativity. It is trivial to check that  $\sigma$  is a ring automorphism of  $R$ , and  $\delta : R \rightarrow R$  is a  $\sigma$ -skew derivation. Hence  $R' := R[F; \sigma, \delta]$  is a Noetherian domain with a basis  $\{K^a g^b h^c F^d | a, b, c \in \mathbb{Z}, d \in \mathbb{Z}_{\geq 0}\}$  by [9, Theorem 1.2.9].

Next, define  $\sigma'$  on  $R'$  via:

$$\sigma'(K^a g^b h^c F^d) = q^{-2a} K^a g^b h^c F^d,$$

(for all integers  $d \geq 0$ , and  $a, b, c \in \mathbb{Z}$ ), and extend  $\sigma'$  by additivity and multiplicativity. One can check that  $\sigma'$  is indeed a ring automorphism of  $R'$ . Define  $\delta'$  on  $R'$  via

$$\delta'(R) \equiv 0, \quad \delta'(F) = \frac{K - K^{-1}g^2}{q - q^{-1}}.$$

Also extend  $\delta'$  to all of  $R'$  by additivity and the following equation:

$$\delta'(ab) := \delta'(a)b + \sigma'(a)\delta'(b), \quad \forall a, b \in R'.$$

One can check that  $\delta'$  is a  $\sigma'$ -skew derivation of  $R'$ . Now by the above results,  $U_{g,h} = R'[E; \sigma', \delta']$  is indeed a Noetherian domain, since  $R'$  is. Moreover,  $U_{g,h}$  has a basis  $\{K^a g^b h^c F^d E^t | a, b, c \in \mathbb{Z}, d, t \in \mathbb{Z}_{\geq 0}\}$ . Since  $K^a g^b h^c F^d E^t = q^{-2ad} F^d K^a g^b h^c E^t$ ,

$$\{F^l K^m g^n h^s E^t | l, t \in \mathbb{Z}_{\geq 0}; m, n, s \in \mathbb{Z}\}$$

is also a basis of  $U_{g,h}$ . This basis is called a PBW basis. □

**Proposition 2.2.** (1)  $U_{g,h}$  is isomorphic to  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}] \otimes U_q(\mathfrak{sl}(2))$  as algebras;

(2)  $U_{g,h}$  is an Auslander regular, Auslander Gorenstein and Tdeg-stable algebra with Gelfand-Kirillov dimension 5.

*Proof.* Define  $E' := g^{-1}E$ ,  $K' := g^{-1}K$ . By Theorem 2.1,

$$\{F^a K'^b g^c h^d E'^t | b, c, d \in \mathbb{Z}, a, t \in \mathbb{Z}_{\geq 0}\}$$

is also a basis of  $U_{g,h}$ . Let  $\varphi(E') = 1 \otimes E$ ,  $\varphi(F) = 1 \otimes F$ ,  $\varphi(K') = 1 \otimes K$ ,  $\varphi(g) = g \otimes 1$ ,  $\varphi(h) = h \otimes 1$  and  $\varphi$  extends by additivity and multiplicativity. One can check that  $\varphi$  is an epimorphism of algebras from  $U_{g,h}$  to  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}] \otimes U_q(\mathfrak{sl}(2))$ . Similarly, define  $\phi(1 \otimes E) = E'$ ,  $\phi(1 \otimes F) = F$ ,  $\phi(1 \otimes K) = K'$ ,  $\phi(g \otimes 1) = g$ ,  $\phi(h \otimes 1) = h$ , and extend  $\phi$  by additivity and multiplicativity. Then  $\phi$  is an epimorphism of algebras from  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}] \otimes U_q(\mathfrak{sl}(2))$  to  $U_{g,h}$ . It is easy to verify that  $\phi \circ \varphi = \text{id}$  and  $\varphi \circ \phi = \text{id}$ . So  $\varphi$  is an isomorphism of algebras.

Let us recall that if the global dimension of a Noetherian ring  $A$ , denoted by  $\text{gldim}(A)$ , is finite, then  $\text{gldim}(A) = \text{injdim}(A)$ , the injective dimension of  $A$ . From [9, Section 7.1.11], one obtains that the right global dimension of a Noetherian algebra  $A$  is equal to  $\text{gldim}(A)$  as well. In [1], H. Bass proved that if  $A$  is a commutative Noetherian ring with a finite injective dimension, then  $A$  is Auslander-Gorenstein. Thus  $\text{gldim}(\mathbb{K}[g^{\pm 1}, h^{\pm 1}, K^{\pm 1}]) = 3$ , and

$$\text{gldim } U_{g,h} \leq \text{gldim}(\mathbb{K}[g^{\pm 1}, h^{\pm 1}, K^{\pm 1}]) + 2 = 5$$

by [9, Theorem 7.5.3]. Hence  $U_{g,h}$  is an Auslander regular and Auslander Gorenstein ring by [3, Theorem 4.2].

Recall that an algebra  $A$  with total quotient algebra  $Q(A)$  is said to be Tdeg-stable if

$$\text{Tdeg}(Q(A)) = \text{Tdeg}(A) = \text{GKdim}(A),$$

where  $\text{GKdim}(A)$  is the Gelfand-Kirillov dimension of  $A$ . By [15, Example 7.1],  $U_q(\mathfrak{sl}(2))$  is Tdeg-stable, and  $\text{GKdim}(U_q(\mathfrak{sl}(2))) = 3$ . Since

$$U_{g,h} \cong \mathbb{K}[g^{\pm 1}, h^{\pm 1}] \otimes U_q(\mathfrak{sl}(2)) \cong U_q(\mathfrak{sl}(2))[g, g^{-1}][h, h^{-1}],$$

$$\text{GKdim}(U_{g,h}) = 2 + \text{GKdim}(U_q(\mathfrak{sl}(2))) = 5,$$

and  $U_{g,h}$  is Tdeg-stable by [15, Theorem 1.1]. □

**Remark 2.3.** (1) Since  $U_{g,h} \cong \mathbb{K}[g^{\pm 1}, h^{\pm 1}] \otimes U_q(\mathfrak{sl}(2))$  as algebras, we call the Hopf algebra  $U_{g,h}$  a double loop quantum enveloping algebra.

(2) Since  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}]$  and  $U_q(\mathfrak{sl}(2))$  are Hopf algebras,  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}] \otimes U_q(\mathfrak{sl}(2))$  has a natural Hopf algebra structure. However, as

$$\Delta(E') = h^{-1}g^{-1} \otimes E' + E' \otimes hK',$$

and

$$\Delta(F) = K'^{-1}hg \otimes F + F \otimes h^{-1},$$

by (1.5) and (1.6), the above isomorphism of algebras is not an isomorphism of Hopf algebras, i.e.,  $U_{g,h}$  has a different coproduct than the usual coproduct of the tensor product of the two coalgebras.

**Corollary 2.4.** *Suppose  $q$  is not a root of unity. Then the center of  $U_{g,h}$  is equal to  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}, C]$ , where  $C = FE + \frac{qK+q^{-1}K^{-1}g^2}{(q-q^{-1})^2}$ .*

*Proof.* Let  $c' = F'E' + \frac{qK'+q^{-1}K'^{-1}}{(q-q^{-1})^2}$ , where  $K'^{\pm}, E', F'$  are the Chevalley generators of  $U_q(\mathfrak{sl}(2))$ . Then the center of  $U_q(\mathfrak{sl}(2))$  is generated by  $c'$  by [6, Theorem VI.4.8]. Since

$$U_{g,h} \cong \mathbb{K}[g^{\pm 1}, h^{\pm 1}] \otimes U_q(\mathfrak{sl}(2))$$

as algebras by Proposition 2.2, the center of  $U_{g,h}$  is isomorphic to  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}] \otimes \mathbb{K}[c']$ . So the center of  $U_{g,h}$  is equal to  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}, c_1]$ , where  $c_1 = g^{-1}FE + \frac{g^{-1}qK+gg^{-1}K^{-1}}{q-q^{-1}}$ . Hence the center of  $U_{g,h}$  is equal to  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}, C]$ , where  $C = FE + \frac{qK+q^{-1}K^{-1}g^2}{(q-q^{-1})^2}$ .  $\square$

The element  $C = FE + \frac{qK+q^{-1}K^{-1}g^2}{(q-q^{-1})^2}$  is called a Casimir element of  $U_{g,h}$ .

**Proposition 2.5.** *There exists an anti-involution  $i$  of  $U_{g,h}$  that acts as the identity on all of  $\mathbb{K}[K^{\pm 1}, g^{\pm 1}, h^{\pm 1}]$ .*

*Proof.* Let  $i(E) = -KF$ ,  $i(F) = -EK^{-1}$ ,  $i(K^{\pm 1}) = K^{\pm 1}$ ,  $i(g^{\pm 1}) = g^{\pm 1}$ ,  $i(h^{\pm 1}) = h^{\pm 1}$ . Extend  $i$  by additivity and multiplicativity. Then  $i$  is an anti-involution of  $U_{g,h}$ , which acts as the identity on all of  $\mathbb{K}[K^{\pm 1}, g^{\pm 1}, h^{\pm 1}]$ .  $\square$

Suppose  $M$  is a finitely generated module over an algebra  $A$ . Then the grade of  $M$ , denoted by  $j(M)$ , is defined to be

$$j(M) := \min\{j \geq 0 \mid \text{Ext}_A^j(M, A) \neq 0\}.$$

Recall that an algebra  $A$  is Cohen-Macaulay if

$$j(M) + \text{GKdim}(M) = \text{GKdim}(A)$$

for every nonzero finitely generated  $A$ -module  $M$ .

**Proposition 2.6.** *The algebra  $U_{g,h}$  is a Cohen-Macaulay algebra with  $\text{gldim } U_{g,h} = 5$ .*

*Proof.* Let  $A = \mathbb{K}[g, h, u, v, K, L][F; \alpha][E; \alpha, \delta]$ , where  $\alpha|_{\mathbb{K}[g, h, u, v]} = \text{id}$ ,  $\alpha(K) = q^2K$ ,  $\alpha(L) = q^{-2}L$ ,  $\alpha(F) = F$ ,  $\delta(F) = \frac{K-L}{q-q^{-1}}$ , and  $\delta(\mathbb{K}[g, h, u, v, K, L]) = 0$ . Then  $A$  is Auslander-regular and Cohen-Macaulay by [2, Lemma II.9.10]. Since

$$U_{g,h} \cong A/(gu - 1, hv - 1, KL - 1),$$

$U_{g,h}$  is Auslander-Gorenstein and Cohen-Macaulay by [2, Lemma II.9.11]. Let  $\mathbb{K}$  be the trivial  $U_{g,h}$ -module defined by  $a \cdot 1 = \varepsilon(a)1$ . Then  $\text{GKdim}(\mathbb{K}) = 0$  and  $\text{gldim}(U_{g,h}) = 5$  by [2, Exercise II.9.D].  $\square$

In the presentation for  $U_{g,h}$  given in Section 1, the generators  $K^{\pm 1}$ , and the generators  $E, F$  play a different role respectively. Similar to [4], we write down an equitable presentation for  $U_{g,h}$  as follows.

**Theorem 2.7.** *The algebra  $U_{g,h}$  is isomorphic to the unital associative  $\mathbb{K}$ -algebra with generators  $x^{\pm 1}, y, z; u^{\pm 1}, v^{\pm 1}$  and the following relations:*

$$(2.1) \quad x^{-1}x = xx^{-1} = 1, \quad u^{-1}u = uu^{-1} = 1, \quad v^{-1}v = vv^{-1} = 1,$$

$$(2.2) \quad ux = xu, \quad uy = yu, \quad uz = zu, \quad uv = vu,$$

$$(2.3) \quad vx = xv, \quad yv = vy, \quad zv = vz,$$

$$(2.4) \quad \frac{qxy - q^{-1}yx}{q - q^{-1}} = 1,$$

$$(2.5) \quad \frac{qzx - q^{-1}xz}{q - q^{-1}} = 1,$$

$$(2.6) \quad \frac{qyz - q^{-1}zy}{q - q^{-1}} = 1.$$

*Proof.* Let  $\mathcal{U}_{u,v}$  be the algebra generated by  $x^{\pm 1}, y, z, u^{\pm 1}, v^{\pm 1}$  satisfying the relations from (2.1) to (2.6). Let us define  $\Phi(x^{\pm 1}) = g^{\mp 1}K^{\pm 1}$ ,  $\Phi(y) = K^{-1}g + F(q - q^{-1})$ ,  $\Phi(z) = K^{-1}g - K^{-1}Eq(q - q^{-1})$ ,  $\Phi(u^{\pm 1}) = g^{\mp 1}$ ,  $\Phi(v^{\pm 1}) = h^{\pm 1}$ , and extend  $\Phi$  by additivity and multiplicativity. Then  $\Phi$  is a homomorphism of algebras from  $\mathcal{U}_{u,v}$  to  $U_{g,h}$ .

Define  $\Psi(K^{\pm 1}) = u^{\mp 1}x^{\pm 1}$ ,  $\Psi(F) = \frac{y-x^{-1}}{q-q^{-1}}$ ,  $\Psi(E) = \frac{1-xz}{(q-q^{-1})qu}$ ,  $\Psi(g) = u^{-1}$ , and  $\Psi(h) = v$ . We extend  $\Psi$  by additivity and multiplicativity. It is routine to check that  $\Psi$  is a homomorphism of algebras from  $U_{g,h}$  to  $\mathcal{U}_{u,v}$ . Since  $\Phi\Psi$  fixes each of the generators  $E, F, K^{\pm 1}, g^{\pm 1}, h^{\pm 1}$  of  $U_{g,h}$ ,  $\Phi\Psi = \text{id}$ . Similarly we can check that  $\Psi\Phi = \text{id}$ . So  $\Phi$  is the inverse of  $\Psi$ .  $\square$

Since  $\mathcal{U}_{u,v}$  is isomorphic to  $U_{g,h}$  as algebras, we can regard  $U_{g,h}$  as an algebra generated by  $x^{\pm 1}, u^{\pm 1}, v^{\pm 1}, y$  and  $z$  with relations (2.1)–(2.6). To make the above algebra isomorphisms  $\Phi, \Psi$  into isomorphisms of Hopf algebras, we only need to define the other operations of the Hopf algebra  $U_{g,h}$  with these new generators as follows:

$$(2.7) \quad \Delta(x^{\pm 1}) = x^{\pm 1} \otimes x^{\pm 1},$$

$$(2.8) \quad \Delta(u^{\pm 1}) = u^{\pm 1} \otimes u^{\pm 1},$$

$$(2.9) \quad \Delta(v^{\pm 1}) = v^{\pm 1} \otimes v^{\pm 1},$$

$$(2.10) \quad \Delta(y) = x^{-1} \otimes (x^{-1} - v^{-1}) + u^{-1}vx^{-1} \otimes (y - x^{-1}) + y \otimes v^{-1},$$

$$(2.11) \quad \Delta(z) = x^{-1} \otimes x^{-1} + uv^{-1}x^{-1} \otimes (z - x^{-1}) + (z - x^{-1}) \otimes v,$$

$$(2.12) \quad \varepsilon(x^{\pm 1}) = \varepsilon(u^{\pm 1}) = \varepsilon(v^{\pm 1}) = 1,$$

$$(2.13) \quad \varepsilon(y) = \varepsilon(z) = 1,$$

and

$$(2.14) \quad S(x^{\pm 1}) = x^{\mp 1}, \quad S(u^{\pm 1}) = u^{\mp 1}, \quad S(v^{\pm 1}) = v^{\mp 1},$$

$$(2.15) \quad S(y) = x - x^{-1}y + u, \quad S(z) = x + u^{-1} - u^{-1}xz.$$

Then one can check that the above isomorphisms  $\Phi, \Psi$  are isomorphisms of Hopf algebras. For example,  $\Delta(\Psi(g^{\mp 1}K^{\pm 1})) = \Delta(x^{\pm 1}) = x^{\pm 1} \otimes x^{\pm 1} = (\Psi \otimes \Psi)\Delta(g^{\mp 1}K^{\pm 1})$ .

3. FINITE-DIMENSIONAL REPRESENTATIONS OF  $U_{g,h}$ 

Let  $q$  be a nonzero element in an algebraically closed field  $\mathbb{K}$  with characteristic zero. Moreover, we assume that  $q$  is not a root of unity. The main purpose of this section is to classify all extensions between two finite-dimensional simple  $U_{g,h}$ -modules. Let us start with a description of the finite-dimensional simple  $U_{g,h}$ -modules.

For any three elements  $\lambda, \alpha, \beta \in \mathbb{K}^\times (= \mathbb{K} \setminus \{0\})$  and any  $U_{g,h}$ -module  $V$ , let

$$V^{\lambda, \alpha, \beta} = \{v \in V \mid Kv = \lambda v, gv = \alpha v, hv = \beta v\}.$$

The  $(\lambda, \alpha, \beta)$  is called a weight of  $V$  if  $V^{\lambda, \alpha, \beta} \neq 0$ . A nonzero vector in  $V^{\lambda, \alpha, \beta}$  is called a weight vector with weight  $(\lambda, \alpha, \beta)$ .

The next result is proved by a standard argument.

**Lemma 3.1.** *We have  $EV^{\lambda, \alpha, \beta} \subseteq V^{q^2\lambda, \alpha, \beta}$  and  $FV^{\lambda, \alpha, \beta} \subseteq V^{q^{-2}\lambda, \alpha, \beta}$ .*

**Definition 3.2.** *Let  $V$  be a  $U_{g,h}$ -module and  $(\lambda, \alpha, \beta) \in \mathbb{K}^{\times 3}$ . A nonzero vector  $v$  of  $V$  is a highest weight vector of weight  $(\lambda, \alpha, \beta)$  if*

$$Ev = 0, \quad Kv = \lambda v, \quad gv = \alpha v, \quad hv = \beta v.$$

*A  $U_{g,h}$ -module  $V$  is a standard cyclic module with highest weight  $(\lambda, \alpha, \beta)$  if it is generated by a highest weight vector  $v$  of weight  $(\lambda, \alpha, \beta)$ .*

**Proposition 3.3.** *Any nonzero finite-dimensional  $U_{g,h}$ -module contains a highest weight vector. Moreover, the endomorphisms induced by  $E$  and  $F$  are nilpotent.*

*Proof.* By Lie's theorem, there is a nonzero vector  $w \in V$  and  $(\mu, \alpha, \beta) \in \mathbb{K}^{\times 3}$  such that

$$Kw = \mu w, \quad gw = \alpha w, \quad hw = \beta w.$$

In fact, there is an elementary and more direct proof as follows. Since  $\mathbb{K}$  is algebraically closed and  $V$  is finite-dimensional, there is a nonzero vector  $v \in V$  such that  $Kv = \mu v$  for some element  $\mu \in \mathbb{K}$ . Moreover  $\mu \in \mathbb{K}^\times$  as  $K$  is invertible. Let

$$V_\mu = \{v \in V \mid Kv = \mu v\} \neq 0.$$

Then  $V_\mu$  is also a finite-dimensional vector space. For any  $v \in V_\mu$ , we have

$$K(gv) = g(Kv) = \mu gv.$$

So  $gv \in V_\mu$  and  $g$  induces a linear transformation on the nonzero finite-dimensional vector space  $V_\mu$ . There is a nonzero vector  $v' \in V_\mu$  such that  $gv' = \alpha v'$  for some nonzero element  $\alpha \in \mathbb{K}$ . Let  $V_{\mu, \alpha} = \{v' \in V_\mu \mid gv' = \alpha v'\}$ . Then  $V_{\mu, \alpha}$  is also a nonzero finite-dimensional linear space. Similarly we can prove that  $h(V_{\mu, \alpha}) \subseteq V_{\mu, \alpha}$  as  $gh = hg$ ,  $hK = Kh$  by (1.2) and (1.3). Hence there exists a nonzero vector  $w \in V_{\mu, \alpha}$  and  $(\mu, \alpha, \beta) \in \mathbb{K}^{\times 3}$  such that

$$Kw = \mu w, \quad gw = \alpha w, \quad hw = \beta w.$$

The proof now follows [6, Proposition VI.3.3], using Lemma 3.1.  $\square$

For any positive integer  $m$ , let  $[m] = \frac{q^m - q^{-m}}{q - q^{-1}}$ , and  $[m]! = [1][2] \cdots [m]$ . Similar to the proof of [6, Lemma VI.3.4], we get the following:

**Lemma 3.4.** *Let  $v$  be a highest weight vector of weight  $(\lambda, \alpha, \beta)$ . Set  $v_p = \frac{1}{[p]!} F^p v$  for  $p > 0$  and  $v_0 = v$ . Then*

$$Kv_p = q^{-2p} \lambda v_p, \quad gv_p = \alpha v_p, \quad Fv_{p-1} = [p]v_p, \quad hv_p = \beta v_p$$

and

$$(3.1) \quad Ev_p = \frac{q^{-(p-1)} \lambda - q^{p-1} \lambda^{-1} \alpha^2}{q - q^{-1}} v_{p-1}.$$

**Theorem 3.5.** (a) *Let  $V$  be a finite-dimensional  $U_{g,h}$ -module generated by a highest weight vector  $v$  of weight  $(\lambda, \alpha, \beta)$ . Then*

- (i)  $\lambda = \varepsilon \alpha q^n$ , where  $\varepsilon = \pm 1$  and  $n$  is the integer defined by  $\dim V = n + 1$ .
- (ii) Setting  $v_p = \frac{1}{[p]!} F^p v$ , we have  $v_p = 0$  for  $p > n$  and in addition the set  $\{v = v_0, v_1, \dots, v_n\}$  is a basis of  $V$ .
- (iii) The operator  $K$  acting on  $V$  is diagonalizable with  $(n + 1)$  distinct eigenvalues

$$\{\varepsilon \alpha q^n, \varepsilon \alpha q^{n-2}, \dots, \varepsilon \alpha q^{-n+2}, \varepsilon \alpha q^{-n}\},$$

and the operators  $g, h$  act on  $V$  by scalars  $\alpha, \beta$  respectively.

- (iv) Any other highest weight vector in  $V$  is a scalar multiple of  $v$  and is of weight  $(\lambda, \alpha, \beta)$ .
- (v) The module is simple.

(b) *Any simple finite-dimensional  $U_{g,h}$ -module is generated by a highest weight vector. Two finite-dimensional  $U_{g,h}$ -modules generated by highest weight vectors of the same weight are isomorphic.*

*Proof.* The proof follows that of [6, Theorem VI.3.5] or [13, Theorem 3.4]. It is omitted here.  $\square$

Let us denote the  $(n + 1)$ -dimensional simple  $U_{g,h}$ -module generated by a highest weight vector  $v$  of weight  $(\varepsilon \alpha q^n, \alpha, \beta)$  in Theorem 3.5 by  $V_{\varepsilon, n, \alpha, \beta}$ . Since  $\mathbb{K}$  is an algebraically closed field, the dimension of a simple module over  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}]$  is equal to one. Any such simple  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}]$ -module is determined by  $g \cdot 1 = \alpha, h \cdot 1 = \beta$ , for  $\alpha, \beta \in \mathbb{K}^\times$ . This simple module is denoted by  $\mathbb{K}_{\alpha, \beta} := \mathbb{K} \cdot 1$  in the sequel. The finite-dimensional simple  $U_q(\mathfrak{sl}(2))$ -modules are characterized in [6, Theorem VI.3.5]. These simple modules are denoted by  $V_{\varepsilon, n}$ , where  $\varepsilon = \pm 1$ , and  $n \in \mathbb{Z}_{\geq 0}$ . By Proposition 2.2 and [8, Proposition 16.1], every finite-dimensional simple  $U_{g,h}$ -module is isomorphic to  $\mathbb{K}_{\alpha, \beta} \otimes V_{\varepsilon, n}$ . It is not difficult to verify that  $\mathbb{K}_{\alpha, \beta} \otimes V_{\varepsilon, n}$  is isomorphic to  $V_{\varepsilon, n, \alpha, \beta}$ .

**Corollary 3.6** (Clebsch-Gordan Formula). *Let  $n \geq m$  be two non-negative integers. There exists an isomorphism of  $U_{g,h}$ -modules*

$$V_{\varepsilon, n, \alpha, \beta} \otimes V_{\varepsilon', m, \alpha', \beta'} \cong V_{\varepsilon \varepsilon', n+m, \alpha \alpha', \beta \beta'} \oplus V_{\varepsilon \varepsilon', n+m-2, \alpha \alpha', \beta \beta'} \oplus \dots \oplus V_{\varepsilon \varepsilon', n-m, \alpha \alpha', \beta \beta'}.$$

*Proof.* Since  $V_{\varepsilon, n, \alpha, \beta} \otimes V_{\varepsilon', m, \alpha', \beta'} \cong \mathbb{K}_{\alpha \alpha', \beta \beta'} \otimes (V_{\varepsilon, n} \otimes V_{\varepsilon', m})$ , and

$$V_{\varepsilon, n} \otimes V_{\varepsilon', m} \cong V_{\varepsilon \varepsilon', n+m} \oplus V_{\varepsilon \varepsilon', n+m-2} \oplus \dots \oplus V_{\varepsilon \varepsilon', n-m}$$

as modules over  $U_q(\mathfrak{sl}(2))$  by [6, Theorem VII.7.1],

$$V_{\varepsilon, n, \alpha, \beta} \otimes V_{\varepsilon', m, \alpha', \beta'} \cong V_{\varepsilon \varepsilon', n+m, \alpha \alpha', \beta \beta'} \oplus V_{\varepsilon \varepsilon', n+m-2, \alpha \alpha', \beta \beta'} \oplus \dots \oplus V_{\varepsilon \varepsilon', n-m, \alpha \alpha', \beta \beta'}.$$

This completes the proof.  $\square$



**Lemma 3.7.** *Let  $m \in \mathbb{N}$ . Then*

$$[E, F^m] = [m]F^{m-1} \frac{q^{-(m-1)}K - q^{m-1}K^{-1}g^2}{q - q^{-1}}.$$

*Proof.* Let  $E', F', K'$  be the Chevalley generators of  $U_q(\mathfrak{sl}(2))$ . Then

$$[E', F'^m] = [m]F'^{m-1} \frac{q^{-(m-1)}K' - q^{m-1}K'^{-1}}{q - q^{-1}},$$

by [6, Lemma VI.1.3]. Substituting  $Eg^{-1}$ ,  $g^{-1}K$ ,  $F$  for  $E', K', F'$  in the above identity respectively, we obtain

$$[E, F^m] = [m]F^{m-1} \frac{q^{-(m-1)}K - q^{m-1}K^{-1}g^2}{q - q^{-1}}.$$

□

Let  $M := \mathbb{K}_{\alpha, \beta} = \mathbb{K} \cdot 1$  be a module over  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}]$ , where  $g \cdot 1 = \alpha$  and  $h \cdot 1 = \beta$ . About the simple modules over the algebra  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}]$ , we have the following

**Proposition 3.8.** *Given two simple  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}]$ -modules  $M := \mathbb{K}_{\alpha, \beta}$  and  $M' := \mathbb{K}_{\alpha', \beta'}$ , if  $M$  is not isomorphic to  $M'$ , then  $\text{Ext}^n(M', M) = 0$  for all  $n \geq 0$ ; if  $M \cong M'$ , then*

$$\text{Ext}^n(M', M) \cong \begin{cases} \mathbb{K}, & n = 0 \\ \mathbb{K}^2, & n = 1 \\ 0, & n \geq 2. \end{cases}$$

*Proof.* Denote the algebra  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}]$  by  $R$ . Construct a projective resolution of the simple  $R$ -module  $\mathbb{K}_{\alpha, \beta}$  as follows:

$$(3.2) \quad 0 \longrightarrow R \xrightarrow{\varphi_2} R^2 \xrightarrow{\varphi_1} R \xrightarrow{\varphi_0} \mathbb{K}_{\alpha, \beta} \longrightarrow 0,$$

where

$$\varphi_0(r(g, h)) = r(\alpha, \beta), \quad \varphi_1(r(g, h), s(g, h)) = r(g, h)(g - \alpha) + s(g, h)(h - \beta),$$

and

$$\varphi_2(r(g, h)) = (r(g, h)(h - \beta), -r(g, h)(g - \alpha))$$

for  $r(g, h), s(g, h) \in R$ . Applying the functor  $\text{Hom}_R(-, \mathbb{K}_{\alpha, \beta})$  to the exact sequence (3.2), we obtain the following complex:

$$(3.3) \quad 0 \longrightarrow \mathbb{K} \xrightarrow{\varphi_1^*} \mathbb{K}^2 \xrightarrow{\varphi_2^*} \mathbb{K} \longrightarrow 0.$$

For any  $\theta \in \text{Hom}_R(R, \mathbb{K}_{\alpha, \beta})$ ,

$$\varphi_1^*(\theta)((1, 0)) = \theta(g - \alpha) = (g - \alpha)\theta(1) = 0,$$

$$\varphi_1^*(\theta)((0, 1)) = \theta(h - \beta) = (h - \beta)\theta(1) = 0.$$

This means that  $\varphi_1^* = 0$ . Similarly, one can prove that  $\varphi_2^* = 0$ . So

$$\text{Ext}^0(\mathbb{K}_{\alpha, \beta}, \mathbb{K}_{\alpha, \beta}) \cong \mathbb{K}, \quad \text{Ext}^1(\mathbb{K}_{\alpha, \beta}, \mathbb{K}_{\alpha, \beta}) \cong \mathbb{K}^2,$$

$$\text{Ext}^2(\mathbb{K}_{\alpha, \beta}, \mathbb{K}_{\alpha, \beta}) \cong \mathbb{K}, \quad \text{Ext}^n(\mathbb{K}_{\alpha, \beta}, \mathbb{K}_{\alpha, \beta}) = 0$$

for  $n \geq 3$ .

If we use the functor  $\text{Hom}_R(-, \mathbb{K}_{\alpha', \beta'})$  to replace the functor  $\text{Hom}_R(-, \mathbb{K}_{\alpha, \beta})$  in the above proof, we can also obtain the complex (3.3). In this case, we have

$$\varphi_1^*(\theta)((1, 0)) = \alpha' - \alpha, \quad \varphi_1^*(\theta)((0, 1)) = \beta' - \beta,$$

and

$$\varphi_2^*(\eta)(a) = (\beta' - \beta)a\eta((1, 0)) - (\alpha' - \alpha)a\eta((0, 1))$$

for  $\theta \in \text{Hom}_R(R, \mathbb{K}_{\alpha', \beta'})$ ,  $\eta \in \text{Hom}_R(R^2, \mathbb{K}_{\alpha', \beta'})$ , and  $a \in R$ . Hence both  $\varphi_1^*$  and  $\varphi_2^*$  are not zero linear mappings provided that either  $\alpha \neq \alpha'$ , or  $\beta \neq \beta'$ . Consequently, the sequence (3.3) is exact in this case. So  $\text{Ext}^n(\mathbb{K}_{\alpha, \beta}, \mathbb{K}_{\alpha', \beta'}) = 0$  for  $n \geq 0$ .  $\square$

It is well-known that the group  $\text{Ext}^1(M', M)$  can be described by short exact sequences. Next, we describe  $\text{Ext}^1(\mathbb{K}_{\alpha, \beta}, \mathbb{K}_{\alpha, \beta})$  by short exact sequences.

Let  $0 \rightarrow \mathbb{K}_{\alpha, \beta} \xrightarrow{\varphi} N \xrightarrow{\psi} \mathbb{K}_{\alpha, \beta} \rightarrow 0$  be an element in  $\text{Ext}^1(\mathbb{K}_{\alpha, \beta}, \mathbb{K}_{\alpha, \beta})$ . Suppose  $\{w_1, w_2\}$  be a basis of  $N$  such that  $\psi(w_2) = 1$  and  $w_1 = \varphi(1)$ . Then  $gw_1 = \alpha w_1$ ,  $hw_1 = \beta w_1$ . Suppose  $gw_2 = \alpha w_2 + xw_1$ . Then

$$\alpha\psi(w_2) = \psi(gw_2) = \alpha\psi(w_2).$$

So  $gw_2 = \alpha w_2 + xw_1$ . Similarly, we can prove  $hw_2 = \beta w_2 + yw_1$ . If  $\{u_1, u_2\}$  is another basis satisfying  $u_1 = \varphi(1) = w_1$  and  $\psi(u_2) = 1$ , then  $u_2 - w_2 = \lambda w_1$  for some  $\lambda \in \mathbb{K}$ . Thus  $gu_2 = \alpha w_2 + xw_1 + \lambda\alpha w_1 = \alpha u_2 + xu_1$ . Similarly, we obtain that  $hu_2 = \beta u_2 + yu_1$ . Hence  $x, y$  are independent of the choice of the bases of  $N$ . So we can use  $M_{x, y}$  to denote the module  $N$ . In the following, we abuse notation and use  $M_{x, y}$  to denote the following exact sequence  $0 \rightarrow \mathbb{K}_{\alpha, \beta} \xrightarrow{\varphi} M_{x, y} \xrightarrow{\psi} \mathbb{K}_{\alpha, \beta} \rightarrow 0$  meanwhile.

Let  $0 \rightarrow \mathbb{K}_{\alpha, \beta} \xrightarrow{\varphi'} M_{x', y'} \xrightarrow{\psi'} \mathbb{K}_{\alpha, \beta} \rightarrow 0$  be another element in  $\text{Ext}^1(\mathbb{K}_{\alpha, \beta}, \mathbb{K}_{\alpha, \beta})$ , and  $\{w'_1, w'_2\}$  be a basis of  $M_{x', y'}$  such that  $w'_1 = \varphi'(1)$ ,  $\psi'(w'_2) = 1$  and

$$\begin{aligned} gw'_1 &= \alpha w'_1, & gw'_2 &= \alpha w'_2 + x'w'_1, \\ hw'_1 &= \beta w'_1, & hw'_2 &= \beta w'_2 + y'w'_1. \end{aligned}$$

Consider the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & \mathbb{K}_{\alpha, \beta} & \xrightarrow{\varphi} & N & \xrightarrow{\psi} & \mathbb{K}_{\alpha, \beta} \longrightarrow 0 \\ & & \mu_1 \downarrow & & \mu_2 \downarrow & & \mu_3 \downarrow \\ 0 & \longrightarrow & \mathbb{K}_{\alpha, \beta} & \xrightarrow{\varphi'} & N' & \xrightarrow{\psi'} & \mathbb{K}_{\alpha, \beta} \longrightarrow 0, \end{array}$$

where  $\mu_i$  are isomorphisms. Then  $\mu_2(w_1) = \mu_2(\varphi(1)) = \varphi'\mu_1(1) = \mu_1(1)w'_1$ . Suppose  $\mu_2(w_2) = aw'_1 + bw'_2$ . Then  $g\mu_2(w_2) = (a\alpha + bx')w'_1 + b\alpha w'_2$ , and

$$\mu_2(gw_2) = \mu_2(\alpha w_2 + xw_1) = (a\alpha + x\mu_1(1))w'_1 + b\alpha w'_2.$$

Since  $g\mu_2(w_2) = \mu_2(gw_2)$ ,  $bx' = x\mu_1(1)$ . Similarly, we have  $by' = y\mu_1(1)$ . Moreover,

$$\mu_3(1) = \mu_3(\psi(w_2)) = \psi'(\mu_2(w_2)) = b.$$

If  $\mu_1(1) = \mu_3(1) = 1$ , then  $b = 1$  and  $(x, y) = (x', y')$ . Thus  $M_{x, y} = M_{x', y'}$  as elements in the group  $\text{Ext}^1(\mathbb{K}_{\alpha, \beta}, \mathbb{K}_{\alpha, \beta})$  if and only if  $(x, y) = (x', y')$ .

From Proposition 3.8, we know that  $\text{Ext}^1(\mathbb{K}_{\alpha,\beta}, \mathbb{K}_{\alpha,\beta})$  is a vector space over  $\mathbb{K}$ . To describe the operations of the vector space  $\text{Ext}^1(\mathbb{K}_{\alpha,\beta}, \mathbb{K}_{\alpha,\beta})$  in the terms of exact sequences, we use  $I$  to denote the ideal of  $R = \mathbb{K}[g^{\pm 1}, h^{\pm 1}]$  generated by  $g - \alpha$  and  $h - \beta$ , i.e.,  $I = R(g - \alpha) + R(h - \beta)$ . Let  $\xi$  be the embedding homomorphism, and  $f$  be the epimorphism of  $R$ -modules from  $R$  to  $\mathbb{K}_{\alpha,\beta}$ , given by

$$f(a(g, h)) = a(\alpha, \beta), \quad a(g, h) \in R.$$

Then we have the following exact sequence of  $R$ -modules:

$$(3.4) \quad 0 \longrightarrow I \xrightarrow{\xi} R \xrightarrow{f} \mathbb{K}_{\alpha,\beta} \longrightarrow 0.$$

Applying the functor  $\text{Hom}_R(-, \mathbb{K}_{\alpha,\beta})$  to the exact sequence (3.4) yields the exact sequence

$$(3.5) \quad \text{Hom}_R(R, \mathbb{K}_{\alpha,\beta}) \xrightarrow{\tau} \text{Hom}_R(I, \mathbb{K}_{\alpha,\beta}) \xrightarrow{\partial} \text{Ext}^1(\mathbb{K}_{\alpha,\beta}, \mathbb{K}_{\alpha,\beta}) \longrightarrow 0.$$

For any exact sequence of  $R$ -modules  $0 \rightarrow \mathbb{K}_{\alpha,\beta} \xrightarrow{\varphi} M_{x,y} \xrightarrow{\psi} \mathbb{K}_{\alpha,\beta} \rightarrow 0$ , and a basis  $\{w_1, w_2\}$  of  $M_{x,y}$  satisfying  $\psi(w_2) = 1$ ,  $w_1 = \varphi(1)$ , define a homomorphism of  $R$ -modules

$$\sigma : R \rightarrow M_{x,y}, \quad \sigma(1) = w_2.$$

Let  $\eta_{x,y}$  be a homomorphism of  $R$ -modules from  $I$  to  $\mathbb{K}_{\alpha,\beta}$ , where

$$(3.6) \quad \eta_{x,y}(a(g, h)(g - \alpha) + b(g, h)(h - \beta)) = xa(\alpha, \beta) + yb(\alpha, \beta),$$

for  $a(g, h), b(g, h) \in R$ . Now, we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & I & \xrightarrow{\xi} & R & \xrightarrow{f} & \mathbb{K}_{\alpha,\beta} & \longrightarrow & 0 \\ & & \eta_{x,y} \downarrow & & \sigma \downarrow & & \text{id} \downarrow & & \\ 0 & \longrightarrow & \mathbb{K}_{\alpha,\beta} & \xrightarrow{\varphi} & M_{x,y} & \xrightarrow{\psi} & \mathbb{K}_{\alpha,\beta} & \longrightarrow & 0. \end{array}$$

It is easy to check that  $M_{x,y}$  is the pushout of  $\eta_{x,y}$  and  $\xi$ . If we use  $M_{kx,ky}$  for any  $k \in \mathbb{K}$  to replace  $M_{x,y}$ , we get a homomorphism  $\eta_{kx,ky}$  from  $I$  to  $\mathbb{K}_{\alpha,\beta}$ . Similarly, we have a homomorphism  $\eta_{x+x',y+y'}$  from  $I$  to  $\mathbb{K}_{\alpha,\beta}$  by using  $M_{x+x',y+y'}$  to replace  $M_{x,y}$ . From the definition of  $\eta_{x,y}$ , one obtains the following:

$$(3.7) \quad \eta_{kx,ky} = k\eta_{x,y}, \quad \eta_{x+x',y+y'} = \eta_{x,y} + \eta_{x',y'}.$$

Define

$$M_{x,y} \boxplus M_{x',y'} = M_{x+x',y+y'}, \quad k \boxplus M_{x,y} = M_{kx,ky},$$

for  $k \in \mathbb{K}$ . Then  $\{M_{x,y} | x, y \in \mathbb{K}\}$  becomes a vector space over  $\mathbb{K}$ . By [12, Theorem 3.4.3], we have a bijection  $\Psi_1$  from  $\{M_{x,y} | x, y \in \mathbb{K}\}$  to  $\text{Ext}^1(\mathbb{K}_{\alpha,\beta}, \mathbb{K}_{\alpha,\beta})$  such that

$$\Psi_1(M_{x,y}) = \partial(\eta_{x,y}) \in \text{Ext}^1(\mathbb{K}_{\alpha,\beta}, \mathbb{K}_{\alpha,\beta}).$$

It follows from (3.7) that

$$\Psi_1(M_{kx,ky}) = k\Psi_1(M_{x,y}), \quad \Psi_1(M_{x+x',y+y'}) = \Psi_1(M_{x,y}) + \Psi_1(M_{x',y'}).$$

Thus  $\Psi_1$  is an isomorphism of vector spaces.

**Proposition 3.9.** *Let  $V_{\varepsilon,n}$  be a simple  $U_q(\mathfrak{sl}(2))$ -module with a basis  $\{v_0, \dots, v_n\}$  satisfying  $E'v_0 = 0, E'v_p = \varepsilon[n-p+1]v_{p-1}, v_p = \frac{F'^p}{[p]!}v_0$ , for  $p = 1, \dots, n$ ;  $F'v_n = 0, K'v_p = \varepsilon q^{n-2p}v_p$  for  $p = 0, \dots, n$ , where  $E', K', F'$  are Chevalley generators of  $U_q(\mathfrak{sl}(2))$ . Then*

$$V_{\varepsilon,n} \otimes M_{x,y} \in \text{Ext}^1(V_{\varepsilon,n,\alpha,\beta}, V_{\varepsilon,n,\alpha,\beta}),$$

where  $M_{x,y} \in \text{Ext}^1(\mathbb{K}_{\alpha,\beta}, \mathbb{K}_{\alpha,\beta})$ . The action of  $U_{g,h}$  on  $V_{\varepsilon,n} \otimes M_{x,y}$  with the basis

$$\{v_0 \otimes w_1, \dots, v_n \otimes w_1; v_0 \otimes w_2, \dots, v_n \otimes w_2\}$$

is given by

$$(3.8) \quad \begin{cases} E(v_0 \otimes w_1) = E(v_0 \otimes w_2) = F(v_n \otimes w_1) = F(v_n \otimes w_2) = 0, \\ E(v_p \otimes w_1) = E'v \otimes gw_1 = \varepsilon[n-p+1]\alpha v_{p-1} \otimes w_1, \\ E(v_p \otimes w_2) = \varepsilon\alpha[n-p+1]v_{p-1} \otimes w_2 + \varepsilon[n-p+1]xv_{p-1} \otimes w_1, \end{cases}$$

for  $p = 1, \dots, n$ ;

$$(3.9) \quad v_p \otimes w_i = \frac{F^p}{[p]!}v_1 \otimes w_i, \quad F(v_n \otimes w_i) = 0$$

for  $p = 1, \dots, n, i = 1, 2$ ;

$$(3.10) \quad \begin{cases} K(v_p \otimes w_1) = K'v_p \otimes gw_1 = \varepsilon\alpha q^{n-2p}(v_p \otimes w_1), \\ K(v_p \otimes w_2) = \varepsilon\alpha q^{n-2p}(v_p \otimes w_2) + \varepsilon q^{n-2p}x(v_p \otimes w_1), \end{cases}$$

for  $p = 0, 1, \dots, n$ ; and

$$(3.11) \quad \begin{cases} g(v_p \otimes w_1) = \alpha(v_p \otimes w_1), \quad g(v_p \otimes w_2) = \alpha(v_p \otimes w_2) + x(v_p \otimes w_1), \\ h(v_p \otimes w_1) = \beta(v_p \otimes w_1), \quad h(v_p \otimes w_2) = \beta(v_p \otimes w_2) + y(v_p \otimes w_1), \end{cases}$$

for  $p = 0, 1, \dots, n$ . Moreover,  $V_{\varepsilon,n} \otimes -$  is an injective linear mapping from the linear space  $\text{Ext}^1(\mathbb{K}_{\alpha,\beta}, \mathbb{K}_{\alpha,\beta})$  to the linear space  $\text{Ext}^1(V_{\varepsilon,n,\alpha,\beta}, V_{\varepsilon,n,\alpha,\beta})$ .

*Proof.* We only need to prove that the mapping  $V_{\varepsilon,n} \otimes -$  is an injective linear mapping, since it is easy to check the other results. Consider the following commutative diagram

$$\begin{array}{ccccccc} 0 & \longrightarrow & V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta} & \xrightarrow{\text{id} \otimes \varphi} & V_{\varepsilon,n} \otimes M_{x,y} & \xrightarrow{\text{id} \otimes \psi} & V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta} \longrightarrow 0 \\ & & \text{id} \downarrow & & \mu \downarrow & & \text{id} \downarrow \\ 0 & \longrightarrow & V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta} & \xrightarrow{\text{id} \otimes \varphi'} & V_{\varepsilon,n} \otimes M_{x',y'} & \xrightarrow{\text{id} \otimes \psi'} & V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta} \longrightarrow 0. \end{array}$$

Since  $(\text{id} \otimes \psi')\mu(v_0 \otimes w_2) = (\text{id} \otimes \psi)(v_0 \otimes w_2) = v_0 \otimes 1 = (\text{id} \otimes \psi')(v_0 \otimes w'_2)$ ,

$$\mu(v_0 \otimes w_2) = v_0 \otimes w'_2 + v \otimes w'_1$$

for some  $v \in V_{\varepsilon,n}$ . Then  $g\mu(v_0 \otimes w_2) = \alpha(v_0 \otimes w'_2 + v \otimes w'_1) + x'v_0 \otimes w'_1$ , and

$$\mu(g(v_0 \otimes w_2)) = \mu(\alpha v_0 \otimes w_2 + xv_0 \otimes w_1) = \alpha(v_0 \otimes w'_2 + v \otimes w'_1) + xv_0 \otimes w'_1.$$

Since  $g\mu(v_0 \otimes w_2) = \mu(g(v_0 \otimes w_2))$ , we have  $x = x'$ . Similarly, we can prove that  $y = y'$ . So  $V_{\varepsilon,n} \otimes -$  induces an injective mapping from  $\text{Ext}^1(\mathbb{K}_{\alpha,\beta}, \mathbb{K}_{\alpha,\beta})$  to  $\text{Ext}^1(V_{\varepsilon,n,\alpha,\beta}, V_{\varepsilon,n,\alpha,\beta})$ .

To prove  $V_{\varepsilon,n} \otimes -$  is linear, we choose an exact sequence of  $U_q(\mathfrak{sl}(2))$ -modules

$$0 \longrightarrow L \longrightarrow P \xrightarrow{f} V_{\varepsilon,n} \longrightarrow 0,$$

where  $P$  is a finitely generated projective  $U_q(\mathfrak{sl}(2))$ -module. Then  $P \otimes R$  is a projective  $U_{g,h}$ -module, and the kernel  $Q$  of  $F$  is  $\text{Ker } f \otimes R + P \otimes I$ , where  $F$  is a homomorphism from  $P \otimes R$  to  $V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta}$  given by  $F(a \otimes b) = f(a) \otimes b \cdot 1$ ,

$$I = R(g - \alpha) + R(h - \beta).$$

Applying the functor  $\text{Hom}_{U_{g,h}}(-, V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta})$  to the exact sequence

$$0 \longrightarrow Q \longrightarrow P \otimes R \xrightarrow{F} V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta} \longrightarrow 0$$

yields an exact sequence

$$(3.12) \quad \text{Hom}_{U_{g,h}}(U_{g,h}, A) \xrightarrow{\tau} \text{Hom}_{U_{g,h}}(Q, A) \xrightarrow{\partial} \text{Ext}^1(A, A) \longrightarrow 0,$$

where  $A = V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta}$ . Define a homomorphism of  $U_{g,h}$ -modules  $\sigma : P \otimes R \rightarrow V_{\varepsilon,n} \otimes M_{x,y}$  by

$$\sigma(a \otimes b) = f(a) \otimes b \cdot w_2, \quad a \otimes b \in P \otimes R,$$

and a homomorphism of  $U_{g,h}$ -modules  $\zeta_{x,y} : Q \rightarrow V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta}$  by

$$\zeta_{x,y}(a \otimes b) = \begin{cases} 0, & a \otimes b \in \text{Ker } f \otimes L, \\ \eta_{x,y}(b)f(a) \otimes 1, & a \otimes b \in P \otimes (R(g - \alpha) + R(h - \beta)), \end{cases}$$

where  $\eta_{x,y}$  is defined by (3.6). Let  $\nu$  be the embedding mapping from  $Q$  to  $P \otimes R$ . Then we have the following commutative diagram:

$$\begin{array}{ccccccccc} 0 & \longrightarrow & Q & \xrightarrow{\nu} & P \otimes R & \xrightarrow{F} & V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta} & \longrightarrow & 0 \\ & & \downarrow \zeta_{x,y} & & \downarrow \sigma & & \downarrow \text{id} & & \\ 0 & \longrightarrow & V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta} & \xrightarrow{1 \otimes \varphi} & V_{\varepsilon,n} \otimes M_{x,y} & \xrightarrow{1 \otimes \psi} & V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta} & \longrightarrow & 0. \end{array}$$

It is easy to check that  $V_{\varepsilon,n} \otimes M_{x,y}$  is the pushout of  $\zeta_{x,y}$  and  $\nu$ . If we use  $M_{kx,ky}$  for any  $k \in \mathbb{K}$  to replace  $M_{x,y}$ , we will obtain a homomorphism  $\zeta_{kx,ky}$  from  $Q$  to  $V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta}$ . Similarly, we get a homomorphism  $\zeta_{x+x',y+y'}$  from  $Q$  to  $V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta}$  by using  $M_{x+x',y+y'}$  to replace  $M_{x,y}$ . From the definitions of these mappings and (3.7), we obtain the following

$$(3.13) \quad \zeta_{kx,ky} = k\zeta_{x,y}, \quad \zeta_{x+x',y+y'} = \zeta_{x,y} + \zeta_{x',y'}.$$

We abuse notation and write  $V_{\varepsilon,n} \otimes M_{x,y}$  for the following exact sequence

$$0 \longrightarrow V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta} \xrightarrow{1 \otimes \varphi} V_{\varepsilon,n} \otimes M_{x,y} \xrightarrow{1 \otimes \psi} V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta} \longrightarrow 0.$$

Define

$$(V_{\varepsilon,n} \otimes M_{x,y}) \boxplus (V_{\varepsilon,n} \otimes M_{x',y'}) = V_{\varepsilon,n} \otimes M_{x+x',y+y'} \quad k \boxdot (V_{\varepsilon,n} \otimes M_{x,y}) = V_{\varepsilon,n} \otimes M_{kx,ky},$$

for  $k \in \mathbb{K}$ . Then  $\{V_{\varepsilon,n} \otimes M_{x,y} | x, y \in \mathbb{K}\}$  becomes a vector space with the above operations. By [12, Theorem 3.4.3], we have an injective linear mapping  $\Psi_2$  of linear spaces from

$$\{V_{\varepsilon,n} \otimes M_{x,y} | x, y \in \mathbb{K}\}$$

to  $\text{Ext}^1(V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta}, V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta})$  such that

$$\Psi_2(V_{\varepsilon,n} \otimes M_{x,y}) = \partial(\zeta_{x,y}) \in \text{Ext}^1(V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta}, V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta}).$$

Therefore,

$$\Psi_2(V_{\varepsilon,n} \otimes M_{kx,ky}) = \partial(k\zeta_{x,y}) = k\Psi_2(V_{\varepsilon,n} \otimes M_{x,y})$$

and

$$\Psi_2(V_{\varepsilon,n} \otimes M_{x+x',y+y'}) = \Psi_2(V_{\varepsilon,n} \otimes M_{x,y}) + \Psi_2(V_{\varepsilon,n} \otimes M_{x',y'})$$

by (3.13). Since  $\Psi_2$  is injective and linear,

$$V_{\varepsilon,n} \otimes k \square M_{x,y} = V_{\varepsilon,n} \otimes M_{kx,ky} = k \square (V_{\varepsilon,n} \otimes M_{x,y})$$

and

$$V_{\varepsilon,n} \otimes (M_{x,y} \boxplus M_{x',y'}) = V_{\varepsilon,n} \otimes M_{x+x',y+y'} = (V_{\varepsilon,n} \otimes M_{x,y}) \boxplus (V_{\varepsilon,n} \otimes M_{x',y'}).$$

By now, we have completed the proof.  $\square$

We now completely classify all extensions between two finite-dimensional simple  $U_{g,h}$ -modules.

**Theorem 3.10.** *Suppose  $q$  is not a root of unity. Given two simple  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}]$ -modules  $\mathbb{K}_{\alpha,\beta}$ ,  $\mathbb{K}_{\alpha',\beta'}$  and a finite-dimensional simple  $U_q(\mathfrak{sl}(2))$ -module  $V_{\varepsilon,n}$ , the assignment  $V_{\varepsilon,n} \otimes -$  is an isomorphism of vector spaces from  $\text{Ext}^1(\mathbb{K}_{\alpha',\beta'}, \mathbb{K}_{\alpha,\beta})$  to  $\text{Ext}^1(M', M)$ . Here,*

$$M := V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta} \cong V_{\varepsilon,n,\alpha,\beta}, \quad M' := V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha',\beta'} \cong V_{\varepsilon,n,\alpha',\beta'}.$$

Moreover,  $\text{Ext}^1(V_{\varepsilon,m,\alpha,\beta}, V_{\varepsilon',n,\alpha',\beta'}) = 0$  provided that  $(\varepsilon, m, \alpha, \beta) \neq (\varepsilon', n, \alpha', \beta')$ .

*Proof.* Let  $C$  be the Casimir element of  $U_{g,h}$  defined in Corollary 2.4,

$$d_{m,n} = \frac{q^{m+1}}{(q^{m-n} - \varepsilon\varepsilon')(q^{m+n+2} - \varepsilon\varepsilon')} \left( C - \varepsilon'\alpha' \frac{q^{n+1} + q^{-n-1}}{(q - q^{-1})^2} \right),$$

and

$$a = \begin{cases} \frac{h-\beta'}{\beta-\beta'}, & \text{if } \beta \neq \beta' \\ \frac{g-\alpha'}{\alpha-\alpha'}, & \text{if } \alpha \neq \alpha' \\ \frac{\varepsilon(q-q^{-1})^2}{\alpha} d_{m,n}, & \text{otherwise.} \end{cases}$$

Then  $a$  is in the center of  $U_{g,h}$  by Corollary 2.4.

Observe that  $V_{\varepsilon,m,\alpha,\beta} \cong V_{\varepsilon',n,\alpha',\beta'}$  if and only if  $\varepsilon = \varepsilon'$ ,  $\alpha = \alpha'$ ,  $\beta = \beta'$ , and  $m = n$ .

Suppose  $V_{\varepsilon,m,\alpha,\beta}$  is not isomorphic to  $V_{\varepsilon',n,\alpha',\beta'}$ , then  $(\varepsilon, m, \alpha, \beta) \neq (\varepsilon', n, \alpha', \beta')$ . Let  $v \in V_{\varepsilon,m,\alpha,\beta}$  be a nonzero highest weight vector satisfying

$$Kv = \varepsilon\alpha q^m v, \quad gv = \alpha v, \quad hv = \beta v, \quad Ev = 0.$$

Then

$$\begin{aligned} d_{m,n}v &= \frac{q^{m+1}}{q^{2m+2} - \varepsilon\varepsilon'q^{m+n+2} - \varepsilon\varepsilon'q^{m-n+1}} \left( FE + \frac{qK + q^{-1}K^{-1}q^2}{(q - q^{-1})^2} - \varepsilon'\alpha \frac{q^{n+1} + q^{-n-1}}{(q - q^{-1})^2} \right)v \\ &= \frac{\varepsilon\alpha}{(q - q^{-1})^2}v, \end{aligned}$$

in the case  $\alpha' = \alpha$ . Therefore  $am = m$  for any  $m \in V_{\varepsilon,m,\alpha,\beta}$  by Schur's Lemma, since  $V_{\varepsilon,m,\alpha,\beta}$  is a simple module and  $a$  induces an endomorphism of  $V_{\varepsilon,m,\alpha,\beta}$ . Similarly, we can prove that  $am = 0$  for any  $m \in V_{\varepsilon',n,\alpha',\beta'}$ .

Consider the short exact sequence of  $U_{g,h}$ -modules

$$(3.14) \quad 0 \longrightarrow V_{\varepsilon,m,\alpha,\beta} \xrightarrow{\phi} V \xrightarrow{\varphi} V_{\varepsilon',n,\alpha',\beta'} \longrightarrow 0.$$

Since  $a\varphi(V) = \varphi(aV) = 0$ ,

$$\phi(V_{\varepsilon,m,\alpha,\beta}) = \text{Ker } \varphi \supseteq aV \supseteq a\phi(V_{\varepsilon,m,\alpha,\beta}) = \phi(aV_{\varepsilon,m,\alpha,\beta}) = \phi(V_{\varepsilon,m,\alpha,\beta}).$$

So  $\phi(V_{\varepsilon,m,\alpha,\beta}) = aV$ . In particular,  $a(av) = av$  for any  $v \in V$ . Therefore

$$V = \text{Ker } a \oplus aV = \text{Ker } a \oplus \phi(V_{\varepsilon,m,\alpha,\beta}).$$

Hence the sequence (3.14) is splitting and  $\text{Ext}^1(V_{\varepsilon',n,\alpha',\beta'}, V_{\varepsilon,m,\alpha,\beta}) = 0$ .

Suppose  $(\alpha, \beta) \neq (\alpha', \beta')$ . Then  $\text{Ext}^1(M', M) = 0$  and  $\text{Ext}^1(\mathbb{K}_{\alpha',\beta'}, \mathbb{K}_{\alpha,\beta}) = 0$  by Proposition 3.8. It is trivial that  $V_{\varepsilon,n} \otimes -$  is an isomorphism of linear spaces.

Next, we assume that  $V_{\varepsilon,n,\alpha',\beta'} \cong V_{\varepsilon,m,\alpha,\beta} \cong V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta}$ . Consider the following exact sequence of  $U_{g,h}$ -modules

$$(3.15) \quad 0 \longrightarrow M \xrightarrow{\phi} V \xrightarrow{\varphi} M \longrightarrow 0.$$

Since  $U_q(\mathfrak{sl}(2))$  is a subalgebra of  $U_{g,h}$ , we can regard the exact sequence (3.15) as a sequence of  $U_q(\mathfrak{sl}(2))$ -modules. Since every finite-dimensional  $U_q(\mathfrak{sl}(2))$ -module is semisimple, there is a homomorphism  $\lambda$  of  $U_q(\mathfrak{sl}(2))$ -modules from  $M$  to  $V$  such that  $\varphi\lambda = \text{id}_M$ . For any  $v \in V$ , we have  $v = (v - \lambda\varphi(v)) + \lambda\varphi(v)$ . Moreover,  $\varphi(v - \lambda\varphi(v)) = 0$ . Hence

$$V = \text{Ker } \varphi \oplus \text{Im } \lambda = \text{Im } \phi \oplus \text{Im } \lambda,$$

where  $\text{Im } \lambda \cong V_{\varepsilon,n}$  as  $U_q(\mathfrak{sl}(2))$ -modules. Let  $K' = Kg^{-1}$ . Suppose  $u_1, u_2$  are the highest weight vectors of the  $U_{g,h}$ -module  $\text{Im } \phi$  and the  $U_q(\mathfrak{sl}(2))$ -module  $\text{Im } \lambda$  respectively. Then

$$\left\{ \frac{F^i}{[i]!} u_1, \frac{F^i}{[i]!} u_2 \mid i = 0, \dots, n \right\}$$

is a basis of  $V$ . Moreover,

$$\begin{aligned} K\varphi(u_2) &= g\varphi(K'u_2) = \varepsilon\alpha q^n \varphi(u_2), & E\varphi(u_2) &= g\varphi(Eg^{-1}u_2) = 0, \\ g\varphi(u_2) &= \alpha\varphi(u_2), & h\varphi(u_2) &= \beta\varphi(u_2). \end{aligned}$$

So  $\varphi(u_2)$  is a highest weight vector of  $M$ . Suppose  $gu_2 = \sum_{i=0}^n a_i \frac{1}{[i]!} F^i u_1 + \sum_{i=0}^n x_i \frac{1}{[i]!} F^i u_2$ .

Then

$$(3.16) \quad \varepsilon q^n gu_2 = gK'u_2 = K'gu_2 = \varepsilon \left( \sum_{i=0}^n q^{n-2i} a_i \frac{1}{[i]!} F^i u_1 + \sum_{i=0}^n q^{n-2i} x_i \frac{1}{[i]!} F^i u_2 \right).$$

Since  $q^m \neq 1$  for any positive integer  $m$ , we obtain  $a_i = x_i = 0$ ,  $i = 1, 2, \dots, n$  from (3.16). Hence  $gu_2 = a_0 u_2 + x_0 u_1$ . Moreover,  $a_0 \varphi(u_2) = \varphi(gu_2) = g\varphi(u_2) = \alpha\varphi(u_2)$ . So  $a_0 = \alpha$ . Similarly, we can prove that  $hu_2 = \beta u_2 + y_0 u_1$ . Moreover, by using Lemma 3.7, one can prove that

$$(3.17) \quad \begin{cases} E(u_1) = E(u_2) = F\left(\frac{1}{[n]!} F^n u_1\right) = F\left(\frac{1}{[n]!} F^n u_2\right) = 0, \\ E\left(\frac{1}{[p]!} F^p u_1\right) = \varepsilon[n-p+1]\alpha F^{p-1} u_1, \\ E\left(\frac{F^p}{[p]!} u_2\right) = \varepsilon\alpha[n-p+1]\frac{F^{p-1}}{[p-1]!} u_2 + \varepsilon[n-p+1]x_0 \frac{F^{p-1}}{[p-1]!} u_1, \end{cases}$$

for  $p = 1, \dots, n$ ;

$$(3.18) \quad K\left(\frac{F^p}{[p]!}u_2\right) = \varepsilon\alpha q^{n-2p}\frac{F^p}{[p]!}u_2 + \varepsilon q^{n-2p}x_0\frac{F^p}{[p]!}u_1,$$

for  $p = 0, 1, \dots, n$ ; and

$$(3.19) \quad \begin{cases} g\left(\frac{F^p}{[p]!}u_1\right) = \alpha\frac{F^p}{[p]!}u_1, & g\left(\frac{F^p}{[p]!}u_2\right) = \alpha\frac{F^p}{[p]!}u_2 + x_0\frac{F^p}{[p]!}u_1, \\ h\left(\frac{F^p}{[p]!}u_1\right) = \beta\frac{F^p}{[p]!}u_1, & h\left(\frac{F^p}{[p]!}u_2\right) = \beta\frac{F^p}{[p]!}u_2 + y_0\frac{F^p}{[p]!}u_1, \end{cases}$$

for  $p = 0, 1, \dots, n$ .

Define  $\tau\left(\frac{F^i}{[i]!}u_j\right) = v_i \otimes w_j$  for  $i = 0, 1, \dots, n; j = 1, 2$ , and extend it by linearity. Comparing the relations from (3.8) to (3.11) in Proposition 3.9 with the above relations from (3.17) to (3.19), we know that  $\tau$  is an isomorphism of  $U_{g,h}$ -modules from  $V$  to  $V_{\varepsilon,n} \otimes M_{x_0,y_0}$ . Hence  $V_{\varepsilon,n} \otimes -$  is an isomorphism of linear spaces by Proposition 3.9.  $\square$

**Remark 3.11.** Since  $\text{Ext}^1(V_{\varepsilon,n}, V_{\varepsilon,n}) = 0$  and  $\text{Ext}^1(V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta}, V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta}) \neq 0$ , the functor  $- \otimes \mathbb{K}_{\alpha,\beta}$  is the zero mapping from  $\text{Ext}^1(V_{\varepsilon,n}, V_{\varepsilon,n})$  to  $\text{Ext}^1(V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta}, V_{\varepsilon,n} \otimes \mathbb{K}_{\alpha,\beta})$ . Hence the functor  $- \otimes \mathbb{K}_{\alpha,\beta}$  does not induce an isomorphism.

Since  $U_{g,h}$  is a Hopf algebra, the dual  $M^*$  of any  $U_{g,h}$ -module  $M$  is still a  $U_{g,h}$  module. For  $a \in U_{g,h}$ ,  $f \in M^*$ , the action of  $a$  on  $f$  is given by

$$(af)(m) := f((Sa)m), \quad m \in M,$$

where  $S$  is the antipode of  $U_{g,h}$ . Next we describe the dual module of a simple module over  $U_{g,h}$ .

**Theorem 3.12.** The dual module  $V_{\varepsilon,n,\alpha,\beta}^*$  of the simple  $U_{g,h}$ -module  $V_{\varepsilon,n,\alpha,\beta}$  is a simple module, and  $V_{\varepsilon,n,\alpha,\beta}^* \cong V_{\varepsilon,n,\alpha^{-1},\beta^{-1}}$ .

*Proof.* By Theorem 3.5, we can assume that the simple module  $V_{\varepsilon,n,\alpha,\beta}$  has a basis  $\{v_0, \dots, v_n\}$  with relations:

$$Kv_p = \varepsilon q^{n-2p}\alpha v_p, \quad gv_p = \alpha v_p, \quad hv_p = \beta v_p$$

for  $p = 0, 1, \dots, n$ ,

$$Fv_n = 0, \quad Ev_0 = 0$$

and

$$Ev_p = \varepsilon \frac{q^{n-(p-1)}\alpha - q^{p-1-n}\alpha}{q - q^{-1}}v_{p-1} = \varepsilon\alpha[n-p+1]v_{p-1}$$

for  $p = 1, \dots, n$ . Let  $\{v_0^*, \dots, v_n^*\}$  be the dual basis of  $\{v_0, \dots, v_n\}$ . Then

$$(Ev_n^*)(v_i) = -v_n^*(EK^{-1}v_i) = -q^{2i-n}[n-i+1]v_n^*(v_{i-1}) = 0$$

for  $i = 1, \dots, n$ , and

$$(Ev_n^*)(v_0) = -v_n^*(EK^{-1}v_0) = -\varepsilon\alpha^{-1}q^{-n}v_n^*(0) = 0.$$

Hence  $E(v_n^*) = 0$ . Since

$$(Kv_n^*)(v_i) = v_n^*(K^{-1}v_i) = q^{2i-n}\varepsilon\alpha^{-1}v_n^*(v_i) = \delta_{ni}q^n\varepsilon\alpha^{-1}$$



for  $i = 0, 1, \dots, n$ ,  $Kv_n^* = \varepsilon\alpha^{-1}q^n v_n^*$ . Similarly, that  $gv_n^* = \alpha^{-1}v_n^*$  follows from

$$(gv_n^*)(v_i) = v_n^*(g^{-1}v_i) = \alpha^{-1}v_n^*(v_i)$$

for  $i = 0, 1, \dots, n$ , and that  $hv_n^* = \beta^{-1}v_n^*$  follows from

$$(hv_n^*)(v_i) = v_n^*(h^{-1}v_i) = \beta^{-1}v_n^*(v_i)$$

for  $i = 0, 1, \dots, n$ . So  $V_{\varepsilon,n,\alpha,\beta}^*$  is a simple  $U_{g,h}$ -module generated by the highest weight vector  $v_n^*$  with weight  $(\varepsilon\alpha^{-1}q^n, \alpha^{-1}, \beta^{-1})$ . Hence  $V_{\varepsilon,n,\alpha,\beta}^* \cong V_{\varepsilon,n,\alpha^{-1},\beta^{-1}}$ .  $\square$

Let  $H$  be a Hopf algebra, and  $H^\circ = \{f \in H^* \mid \ker f \text{ contains an ideal } I \text{ such that the dimension of } H/I \text{ is finite}\}$ . Then  $H^\circ$  is a Hopf algebra, which is called the finite dual Hopf algebra of  $H$ . Now let  $M$  be a left module over the Hopf algebra  $U_{g,h}$ . For any  $f \in M^*$  and  $v \in M$ , define a coordinate function  $c_{f,v}^M \in U_{g,h}^*$  via

$$c_{f,v}^M(x) = f(xv) \quad \text{for } x \in H.$$

If  $M$  is finite dimensional, then  $c_{f,v}^M \in U_{g,h}^\circ$ , the finite dual Hopf algebra of  $U_{g,h}$ . The coordinate space  $C(M)$  of  $M$  is a linear subspace of  $U_{g,h}^*$ , spanned by the coordinate functions  $c_{f,v}^M$  as  $f$  runs over  $M^*$  and  $v$  over  $M$ .

**Corollary 3.13.** *Let  $A$  be the subalgebra of  $U_{g,h}^\circ$  generated by all the coordinate functions of all finite dimensional simple  $U_{g,h}$ -modules. Then  $A$  is a sub-Hopf algebra of  $U_{g,h}^\circ$ .*

*Proof.* Let  $\hat{\mathcal{C}}$  be the subcategory of the left  $U_{g,h}$ -module category consisting of all finite direct sums of finite dimensional simple  $U_{g,h}$ -modules. Then  $\hat{\mathcal{C}}$  is closed under tensor products and duals by Corollary 3.6 and Theorem 3.12. Thus  $A$  is a sub-Hopf algebra of  $U_{g,h}^\circ$  and is the directed union of the coordinate spaces  $C(V)$  for  $V \in \hat{\mathcal{C}}$  by [2, Corollary I.7.4].  $\square$

Finally, we describe the simple modules over  $U_{g,h}$  when  $q$  is a root of unity. Assume that the order of  $q$  is  $d > 2$  and define

$$e = \begin{cases} d, & \text{if } d \text{ is odd,} \\ \frac{d}{2}, & \text{otherwise.} \end{cases}$$

We will use the notations  $V(\lambda, a, b)$ ,  $V(\lambda, a, 0)$ ,  $\tilde{V}(\pm q^{1-j}, c)$  to denote finite-dimensional simple  $U_q(\mathfrak{sl}(2))$ -modules. These simple modules have been described in [6, Theorem VI.5.5]. The next results follow from [6, Proposition VI.5.1, Proposition VI.5.2, Theorem VI.5.5] and [8, Proposition 16.1].

**Proposition 3.14.** *Suppose  $q$  is a root of unity. Then*

(1) *Any simple  $U_{g,h}$ -module of dimension  $e$  is isomorphic to a module of the following list:*

(i)  $\mathbb{K}_{\alpha,\beta} \otimes V(\lambda, a, b)$ , where  $\mathbb{K}_{\alpha,\beta} = \mathbb{K} \cdot 1$  is a one-dimensional module over  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}]$ , and  $g \cdot 1 = \alpha$ ,  $h \cdot 1 = \beta$  for some  $\alpha, \beta \in \mathbb{K}^\times$ .

(ii)  $\mathbb{K}_{\alpha,\beta} \otimes V(\lambda, a, 0)$ , where  $\lambda$  is not of the form  $\pm q^j$  for any  $1 \leq j \leq e-1$ ,

(iii)  $\mathbb{K}_{\alpha,\beta} \otimes \tilde{V}(\pm q^{1-j}, c)$ .

(2) *Any simple  $U_{g,h}$ -module of dimension  $n < e-1$  is isomorphic to a module of the form  $V_{\varepsilon,n,\alpha,\beta}$ , where the structure of  $V_{\varepsilon,n,\alpha,\beta}$  is given by Theorem 3.5.*

(3) The dimension of any simple  $U_{g,h}$ -module is not larger than  $e$ .

#### 4. VERMA MODULES AND THE CATEGORY $\mathcal{O}$

In this section, we assume that the nonzero element  $q \in \mathbb{K}$  is not a root of unity. We will study the BGG subcategory of the category of all left  $U_{g,h}$ -modules. For the undefined terms in this section, we refer the reader to [8] and [10].

If  $M$  is a  $U_{g,h}$ -module, a maximal weight vector is any nonzero  $m \in M$  that is killed by  $E$ , and is a common eigenvector for  $K, g, h$ . A standard cyclic module is one which is generated by exactly one maximal weight vector. For each  $(a, b, c) \in \mathbb{K}^{\times 3}$ , define the Verma module

$$V(a, b, c) := U_{g,h}/I(a, b, c),$$

where  $I(a, b, c)$  is the left ideal of  $U_{g,h}$  generated by  $E, K - a, g - b, h - c$ .  $V(a, b, c)$  is a free  $\mathbb{K}[F]$ -module of rank one, by the PBW Theorem 2.1 for  $U_{g,h}$ . Hence the set  $W(V(a, b, c))$  of weights of the Verma module  $V(a, b, c)$  is equal to  $\{(q^{-2n}a, b, c) | n \geq 0\}$ .

About the extension group  $\text{Ext}^1(V(a', b', c'), V(a, b, c))$  of two Verma modules  $V(a', b', c')$ ,  $V(a, b, c)$ , we have the following:

**Proposition 4.1.** *Suppose  $V(a, b, c)$  and  $V(a', b', c')$  are two Verma modules. Then  $\text{Ext}^1((V(a, b, c), V(a, b, c)) \neq 0$  and  $\text{Ext}^1((V(a', b', c'), V(a, b, c)) = 0$  if  $a, b, c; a', b', c'$  satisfy one of the following conditions.*

- (1)  $(b, c) \neq (b', c')$ ;
- (2)  $(b, c) = (b', c')$ ,  $a \neq a'$  and  $aa' \neq q^{-2}b^2$ .

*Proof.* Let  $M_{x,y} \in \text{Ext}^1(\mathbb{K}_{b,c}, \mathbb{K}_{b,c})$  be the module described in Proposition 3.9, where either  $x \neq 0$  or  $y \neq 0$ . Consider the  $U_{g,h}$ -module  $M = V(ab^{-1}) \otimes M_{x,y}$ , where  $V(ab^{-1})$  is a Verma module over  $U_q(\mathfrak{sl}(2))$  generated by a highest weight vector  $v$  with weight  $ab^{-1}$ . Suppose  $w_1, w_2$  is a basis of  $M_{x,y}$  such that  $gw_1 = bw_1$ ,  $gw_2 = bw_2 + xw_1$ ,  $hw_1 = cw_1$ ,  $hw_2 = cw_2 + yw_1$ . Then  $K(v \otimes w_1) = a(v \otimes w_1)$  and

$$K(v \otimes w_2) = a(v \otimes w_2) + ab^{-1}x(v \otimes w_1).$$

Therefore the subspace  $V_1$  of  $M$  generated by

$$\frac{F^n}{[n]!}v \otimes w_1, \quad n \in \mathbb{Z}_{\geq 0}$$

is a  $U_{g,h}$ -module, which is isomorphic to  $V(a, b, c)$ . Moreover  $M/V_1$  is also isomorphic to  $V(a, b, c)$ . Thus  $M \in \text{Ext}^1(V(a, b, c), V(a, b, c))$ . Suppose  $M \cong V(a, b, c) \oplus V(a, b, c)$ . Then the actions of  $g, h$  on  $M$  are given via multiplications by  $b, c$  respectively. This is impossible when either  $x \neq 0$ , or  $y \neq 0$ . So  $M$  is a nonzero element in  $\text{Ext}^1(V(a, b, c), V(a, b, c))$ .

Now let

$$u = \begin{cases} \frac{g-b'}{b-b'}, & \text{if } b \neq b' \\ \frac{h-c'}{c-c'}, & \text{if } c \neq c' \\ \frac{aa'(q-q^{-1})^2}{(a-a')(qaa'-q^{-1}b^2)}(C - \frac{qa'+q^{-1}a'^{-1}b^2}{(q-q^{-1})^2}), & \text{if } a \neq a', aa' \neq q^{-2}b^2, (b, c) = (b', c'), \end{cases}$$

where  $C$ , which is given in Corollary 2.4, is the Casimir element of  $U_{g,h}$ . Then  $u$  is in the center of  $U_{g,h}$  by Corollary 2.4. Suppose  $V(a, b, c)$  and  $V(a', b', c')$  are generated by the

highest weight vectors  $v, v'$  respectively. It is easy to check that  $uv = v$  and  $uv' = 0$ . So  $u$  induces the identity endomorphism of  $V(a, b, c)$  and the zero endomorphism of  $V(a', b', c')$ . Similar to the proof of Theorem 3.10, we can prove that every short exact sequence

$$0 \rightarrow V(a, b, c) \rightarrow N \rightarrow V(a', b', c') \rightarrow 0$$

is splitting. Hence  $\text{Ext}^1((V(a', b', c'), V(a, b, c))) = 0$ .  $\square$

**Remark 4.2.** *It is unknown whether  $\text{Ext}^1(V(q^{-2}a^{-1}b^2, b, c), V(a, b, c)) = 0$  in the case when  $b^2 \neq q^2a^2$ .*

The proof of the following proposition is standard (see e.g. [5], [7] or [8]).

**Proposition 4.3.** (1) *The Verma module  $V(a, b, c)$  has a unique maximal submodule  $N(a, b, c)$ , and the quotient  $V(a, b, c)/N(a, b, c)$  is a simple module  $L(a, b, c)$ .*

(2) *Any standard cyclic module is a quotient of some Verma module.*

By [8, Theorem 4.2] and Proposition 2.2, every Verma module over  $U_{g,h}$  is isomorphic to  $V(\lambda) \otimes \mathbb{K}_{b,c}$ , where  $V(\lambda)$  is a Verma module over  $U_q(\mathfrak{sl}(2))$ . Conversely,  $V(\lambda) \otimes \mathbb{K}_{b,c}$  is a Verma  $U_{g,h}$  module if  $V(\lambda)$  is a Verma module over  $U_q(\mathfrak{sl}(2))$ . In the following, we determine when the Verma module  $V(\lambda) \otimes \mathbb{K}_{b,c}$  is isomorphic to the Verma module  $V(a, b, c)$ , using the isomorphism in Proposition 2.2(1).

**Proposition 4.4.** *Suppose  $V(\lambda)$  is a Verma module over  $U_q(\mathfrak{sl}(2))$  and  $\mathbb{K}_{b,c}$  is a simple module over  $\mathbb{K}[g^{\pm 1}, h^{\pm 1}]$ . Then  $V(\lambda) \otimes \mathbb{K}_{b,c}$  is a Verma module over  $U_{g,h}$  with the highest weight  $(b\lambda, b, c)$ . Conversely, every Verma module  $V(a, b, c)$  over  $U_{g,h}$  is isomorphic to*

$$V(ab^{-1}) \otimes \mathbb{K}_{b,c},$$

where  $V(ab^{-1})$  is a Verma module over  $U_q(\mathfrak{sl}(2))$ .

Therefore the Verma module  $V(a, b, c)$  is isomorphic to  $V(\lambda) \otimes \mathbb{K}_{b',c'}$  if and only if  $(a, b, c) = (b'\lambda, b', c')$ .

*Proof.* Suppose  $E', K', F'$  are Chevalley generators of  $U_q(\mathfrak{sl}(2))$ . Let  $V(\lambda)$  be a Verma module over  $U_q(\mathfrak{sl}(2))$ . Then  $V(\lambda)$  has a basis  $\{v_p | p \in \mathbb{Z}_{\geq 0}\}$  satisfying

$$K'v_p = \lambda q^{-2p}v_p, \quad K'^{-1}v_p = \lambda^{-1}q^{2p}v_p,$$

$$E'v_{p+1} = \frac{q^{-p}\lambda - q^p\lambda^{-1}}{q - q^{-1}}v_p, \quad F'v_p = [p+1]v_{p+1}$$

and  $E'v_0 = 0$ . Since  $U_{g,h} \cong U_q(\mathfrak{sl}(2)) \otimes \mathbb{K}[g^{\pm 1}, h^{\pm 1}]$ , then  $V(\lambda) \otimes \mathbb{K}_{b,c}$  is a cyclic module with the highest vector  $v_0 \otimes 1$ , where the action of  $x \otimes y \in U_q(\mathfrak{sl}(2)) \otimes \mathbb{K}[g^{\pm 1}, h^{\pm 1}]$  on  $v \otimes 1 \in V(\lambda) \otimes \mathbb{K}_{b,c}$  is given by

$$(x \otimes y) \cdot (v \otimes 1) = x \cdot v \otimes y \cdot 1.$$

The highest weight of  $V(\lambda) \otimes \mathbb{K}_{b,c}$  is  $(b\lambda, b, c)$ . Let  $v = 1 + I(b\lambda, b, c)$  be the highest weight vector of the Verma module  $V(b\lambda, b, c)$ . Define a linear map  $f$  from  $V(\lambda) \otimes \mathbb{K}_{b,c}$  to  $V(a, b, c)$  by  $f(v_p \otimes 1) = \frac{1}{[p]!} F^p v$ . Similar to [6, Proposition VI.3.7], we can prove that  $f$  is a homomorphism of  $U_{g,h}$ -modules. Therefore  $V(\lambda) \otimes \mathbb{K}_{b,c}$  is the Verma module with highest weight  $(b\lambda, b, c)$  by Proposition 4.3(2).

Conversely, let  $\lambda = ab^{-1}$ . Consider an infinite-dimensional vector space  $V(\lambda)$  with basis  $\{v_i | i \in \mathbb{Z}_{\geq 0}\}$ . For  $p \geq 0$ , set

$$K'v_p = \lambda q^{-2p}v_p, \quad K'^{-1}v_p = \lambda^{-1}q^{2p}v_p,$$

$$E'v_{p+1} = \frac{q^{-p}\lambda - q^p\lambda^{-1}}{q - q^{-1}}v_p, \quad F'v_p = [p+1]v_{p+1}$$

and  $E'v_0 = 0$ , where  $E', K', F'$  are Chevalley generators of  $U_q(\mathfrak{sl}(2))$ . Then  $V(\lambda)$  is a Verma module over  $U_q(\mathfrak{sl}(2))$  with the above actions by [6, Lemma VI.3.6]. The highest weight of  $V(\lambda) \otimes \mathbb{K}_{b,c}$  is  $(a, b, c)$ . Therefore  $V(\lambda) \otimes \mathbb{K}_{b,c}$  is isomorphic to the Verma module over  $U_{g,h}$  with highest weight  $(a, b, c)$ .  $\square$

One of the basic questions about a Verma module is to determine its maximal weight vectors. We now answer this question.

**Theorem 4.5.** *Let  $V(a, b, c), V(a', b', c')$  be two Verma modules, where  $a, b, c; a', b', c' \in \mathbb{K}^\times$ .*

(1) *If  $V(a, b, c)$  has a maximal weight vector of weight  $(q^{-2n}a, b, c)$ , then it is unique up to scalars and  $a = \varepsilon bq^{n-1}$  with  $n > 0$ .*

(2)  *$\dim_{\mathbb{K}} \text{Hom}_{U_{g,h}}(V(a', b', c'), V(a, b, c)) = 0$  or 1 for all  $(a', b', c')$  and  $(a, b, c)$ , and all nonzero homomorphisms between two Verma modules are injective.*

(3) *The nonzero submodule of  $V(a, b, c)$  (which is unique if it exists) is precisely of the form*

$$V(q^{-2n}a, b, c) = \mathbb{K}[F]v_{q^{-2n}a, b, c}.$$

*Proof.* Suppose  $p(F) = (a_n F^n + a_{n-1} F^{n-1} + \cdots + a_0) \bar{1}$  is a maximal weight vector, where  $\bar{1}$  is the maximal weight vector of  $V(a, b, c)$  and  $a_n \neq 0$ . Then

$$E(p(F)) = [n] \frac{q^{-n+1}a - q^{n-1}a^{-1}b^2}{q - q^{-1}} a_n F^{n-1} \bar{1} + (\text{lower degree terms}) \bar{1} = 0,$$

by Lemma 3.7. This implies  $a = \varepsilon bq^{n-1}$ . Moreover  $p(X) = a_n F^n \bar{1}$  and (1) follows.

(2) follows from (1) and the fact that  $\mathbb{K}[F]$  is a principal ideal domain directly.

If  $M$  is a nonzero submodule of  $V(a, b, c)$ , then  $M$  contains a vector of the highest possible weight  $(q^{-2n}a, b, c)$ . We claim that  $M = V(q^{-2n}a, b, c) = \mathbb{K}[F]v_{q^{-2n}a, b, c}$ , where  $v_{q^{-2n}a, b, c}$  is the weight vector in  $M$  with weight  $(q^{-2n}a, b, c)$ . The weight vector  $v_{q^{-2n}a, b, c}$  is unique up to scalar by (1). To prove the above claim, we only need to show that  $M \subseteq \mathbb{K}[F]v_{q^{-2n}a, b, c}$ .

Suppose, to the contrary, that  $v \in M$  is of the form

$$v = p(F)v_{q^{-2n}a, b, c} + a_{n-1}F^{n-1}\bar{1} + \cdots + a_1F\bar{1} + a_0\bar{1}.$$

We may assume that  $p(F) = 0$  because  $v_{q^{-2n}a, b, c} \in M$ . Since  $K^i v \in M$  for any  $i$ ,  $a_{n-k}F^{n-k}\bar{1} \in M$ ,  $k = 1, 2, \dots, n$ . This is a contradiction since  $(q^{-2i}a, b, c)$  is not a weight of  $M$  if  $i < n$ .  $\square$

**Remark 4.6.** (1) *If  $\frac{a}{b} \neq \varepsilon q^n$  for any  $n \geq 1$ , then the Verma module is a simple module by Theorem 4.5.*

(2) It is well-known that the Verma module  $V(\lambda)$  over  $U_q(\mathfrak{sl}(2))$  is simple provided that  $\lambda \neq \varepsilon q^n$  for any integer  $n > 0$ , where  $\varepsilon = \pm 1$ . Since

$$V(\lambda) \otimes \mathbb{K}_{b,c} \cong V(b\lambda, b, c)$$

by Proposition 4.4,  $V(\lambda) \otimes \mathbb{K}_{b,c}$  is a simple  $U_{g,h}$ -module provided that  $\lambda \neq \varepsilon q^n$  for any  $n$ , where  $\varepsilon = \pm 1$ .

(3) The simple module  $L(a, b, c)$  is finite-dimensional if and only if the only maximal submodule  $N(a, b, c)$  of  $V(a, b, c)$  is equal to  $V(q^{-2n}a, b, c)$  and  $a = \varepsilon bq^{n-1}$  for some  $n \in \mathbb{N}$ . In this case,  $L(\varepsilon bq^{n-1}, b, c) \cong V_{\varepsilon, n-1, b, c}$ , which is given by Theorem 3.5.

Finally, we study the BGG category  $\mathcal{O}$ , which is defined below.

**Definition 4.7.** The BGG category  $\mathcal{O}$  consists of all finitely generated  $U_{g,h}$ -modules and all homomorphisms of modules with the following properties:

- (1) The actions of  $K, g, h$  are diagonalized with finite-dimensional weight spaces.
- (2) The  $B_+$ -action is locally finite, where  $B_+$  is the subalgebra generated by  $E, K^{\pm 1}, g^{\pm 1}, h^{\pm 1}$ .

It is obvious that every Verma module is in  $\mathcal{O}$ . By Theorem 3.5, all finite-dimensional simple  $U_{g,h}$ -modules are in  $\mathcal{O}$ . Any simple module in  $\mathcal{O}$  is isomorphic to either a simple Verma module or a finite-dimensional simple module  $V_{\varepsilon, n, \alpha, \beta}$  described in Theorem 3.5. In fact, if  $M$  is a simple module in  $\mathcal{O}$ , then  $M = U_{g,h}v$  for some common eigenvector  $v$  of  $K, g, h$ . Suppose  $Kv = \lambda v$ . Then  $KE^n v = q^{2n}\lambda E^n v$  for any positive integer  $n$ . Since the action of  $E$  is locally finite, there is an  $n$  such that  $E^n v \neq 0$  and  $E^{n+1}v = 0$ . Thus  $M = U_{g,h}E^n v$  is a standard cyclic  $U_{g,h}$ -module. So it is a quotient of a Verma module. Hence it is isomorphic to either a simple Verma module or a finite-dimensional simple module  $V_{\varepsilon, n, \alpha, \beta}$ .

Suppose  $0 \rightarrow V_{\varepsilon, n, \alpha, \beta} \rightarrow M \rightarrow V_{\varepsilon, n, \alpha, \beta} \rightarrow 0$  is a nonzero element in  $\text{Ext}^1(V_{\varepsilon, n, \alpha, \beta}, V_{\varepsilon, n, \alpha, \beta})$ . We remark that this  $M$  is not in  $\mathcal{O}$  since the actions of  $g, h$  on  $M$  can not be diagonalized by Proposition 3.8 and Theorem 3.10. Similarly, if  $0 \rightarrow V(a, b, c) \rightarrow M \rightarrow V(a, b, c) \rightarrow 0$  is a nonzero element in  $\text{Ext}^1(V(a, b, c), V(a, b, c))$ , then  $M$  is not in  $\mathcal{O}$ .

By using results in [8], we obtain that every finite-dimensional module in  $\mathcal{O}$  is semisimple. In the following we give a direct proof of this fact.

**Proposition 4.8.** Every finite-dimensional module in  $\mathcal{O}$  is semisimple.

*Proof.* Let  $0 = M_0 \subseteq M_1 \subseteq \cdots \subseteq M_n = M$  be a composition series of a finite-dimensional module  $M$  for  $M \in \mathcal{O}$ . We prove that  $M$  is semisimple by using induction. If  $n = 2$ , then we have the following exact sequence

$$0 \rightarrow M_1 \rightarrow M \rightarrow M/M_1 \rightarrow 0.$$

Suppose the above sequence is not splitting, then either the action of  $g$  or the action of  $h$  on  $M$  is not semisimple by Theorem 3.10. Thus  $M \notin \mathcal{O}$ . This contradiction implies that  $M$  is semisimple. Suppose  $M$  is semisimple in the case when  $n = k \geq 2$ . Now let  $n = k+1$ . Then  $M_k = \bigoplus_{i=1}^k S_i$  is a direct sum of simple  $U_{g,h}$ -modules  $S_i$  by the assumption. Now let  $N_i = S_1 \oplus \cdots \oplus \widehat{S_i} \oplus \cdots \oplus S_k$ , where  $\widehat{S_i}$  means that  $S_i$  is omitted. Consider the

following commutative diagrams for  $i = 1, 2, \dots, k$ :

$$\begin{array}{ccccccc} 0 & \longrightarrow & M_k & \xrightarrow{\phi} & M & \xrightarrow{\pi} & M/M_k \longrightarrow 0 \\ & & \lambda_i \downarrow & & \pi_i \downarrow & & \text{id} \downarrow \\ 0 & \longrightarrow & N_i & \xrightarrow{\varphi_i} & M/S_i & \xrightarrow{\psi_i} & M/M_k \longrightarrow 0, \end{array}$$

where  $\phi, \varphi_i$  are embedding mappings, and  $\lambda_i, \pi_i, \pi, \psi_i$  are the canonical projections. Since the bottom exact sequences are splitting by the inductive assumption, there are homomorphisms  $\xi_i : M/S_i \rightarrow N_i$  such that  $\xi_i \varphi_i = \text{id}_{N_i}$ . Define  $\xi : M \rightarrow M_k$  via

$$\xi(m) = \frac{1}{k-1} \sum_{i=1}^k \xi_i \pi_i(m)$$

for  $m \in M$ . Now let  $m = m_1 + \dots + m_k \in M_k$ , where  $m_i \in S_i$ . Then

$$\xi_i \pi_i(m) = \xi_i \pi_i \phi(m) = \xi_i \varphi_i \lambda_i(m) = m - m_i,$$

and

$$\xi \phi(m) = \frac{1}{k-1} \sum_{i=1}^k \xi_i \pi_i(m) = m.$$

This means that the top exact sequence of the above commutative diagrams is splitting. Hence  $M \cong M_k \oplus \text{Ker } \xi \cong M_k \oplus M/M_k \cong S_1 \oplus \dots \oplus S_k \oplus M/M_k$  is semisimple.  $\square$

By the PBW Theorem 2.1, the algebra  $U_{g,h}$  has a triangular decomposition  $\mathbb{K}[F] \otimes H \otimes \mathbb{K}[E]$ , where  $H = \mathbb{K}[K^{\pm 1}, g^{\pm 1}, h^{\pm 1}]$ . In the same way as [8, Definition 11.1], we can define the Harish-Chandra projection  $\xi$  as follows:

$$\xi := \varepsilon \otimes \text{id} \otimes \varepsilon : U_{g,h} = \mathbb{K}[F] \otimes H \otimes \mathbb{K}[E] \rightarrow H.$$

Let  $V(a, b, c)$  be a Verma module generated by a nonzero highest weight vector  $v$ . Then

$$(4.1) \quad Cv = \frac{qa + q^{-1}a^{-1}b^2}{(q - q^{-1})^2}v, \quad gv = bv, \quad hv = cv,$$

where  $C$  is the Casimir element of  $U_{g,h}$ . By Corollary 2.4, the center of  $U_{g,h}$  is  $\mathbb{K}[C, g^{\pm 1}, h^{\pm 1}]$ . For any element  $z \in \mathbb{K}[C, g^{\pm 1}, h^{\pm 1}]$ ,  $zv = \xi_{(a,b,c)}(z)v$  for some  $\xi_{(a,b,c)}(z) \in \mathbb{K}$ . Then

$$\xi_{(a,b,c)} \in \text{Hom}_{\text{alg}}(\mathbb{K}[C, g^{\pm 1}, h^{\pm 1}], \mathbb{K}).$$

We call  $\xi_{(a,b,c)}$  the central character determined by  $V(a, b, c)$ .

**Proposition 4.9.** (1) Suppose  $V(a, b, c)$  and  $V(a', b', c')$  are two Verma modules. Then  $\xi_{(a', b', c')} = \xi_{(a, b, c)}$  if and only if

$$(4.2) \quad (a - a')(aa' - q^{-2}b^2) = 0, \quad b = b', \quad c = c'.$$

(2)  $\text{Hom}_{U_{g,h}}(V(a, b, c), V(a', b', c')) \neq 0$  if and only if  $a = \varepsilon q^{-n-1}b$  and  $a' = \varepsilon q^{n-1}b$  for some nonnegative integer  $n$  and  $(b, c) = (b', c')$ .

*Proof.* Let  $v, v'$  be the nonzero highest weight vectors of  $V(a, b, c)$  and  $V(a', b', c')$  respectively. Then  $\xi_{(a', b', c')} = \xi_{(a, b, c)}$  if and only if  $Cv' = \xi_{(a, b, c)}(C)v'$ ,  $gv' = \xi_{(a, b, c)}(g)v'$  and  $hv' = \xi_{(a, b, c)}(h)v'$ . Thus (4.2) follows from (4.1).

If there is a nonzero homomorphism  $\varphi$  from  $V(a, b, c)$  to  $V(a', b', c')$ , then

$$\xi_{(a', b', c')} = \xi_{(a, b, c)}.$$

Thus (4.2) holds. Suppose  $\varphi(v) = (\sum_{i=0}^n a_i F^i)v'$ , where  $a_n \neq 0$ . Since  $\varphi(Kv) = K\varphi(v)$ ,

$$aa_i = q^{-2i}a_i a'$$

for  $i = 0, 1, \dots, n$ . Hence  $a = q^{-2n}a'$  and  $a_i = 0$  for  $0 \leq i \leq n-1$ . Observe that

$$0 = \varphi(Ev) = E\varphi(v) = a_n EF^n v' = a_n [n] \frac{a' q^{-n+1} - a'^{-1} q^{n-1} b^2}{q - q^{-1}} F^{n-1} v'.$$

Hence  $aa' = q^{2n-2}b^2$ . So  $a' = \varepsilon q^{n-1}b$  and  $a = \varepsilon q^{-n-1}b$ .

Conversely, notice that  $V(a, b, c) = \mathbb{K}[F]v$  and  $V(a', b', c') = \mathbb{K}[F]v'$  are two free  $\mathbb{K}[F]$ -modules. Thus the mapping

$$\varphi(f(F)v) = f(F)F^n v', \quad f(F) \in \mathbb{K}[F]$$

is a nonzero linear mapping. Since  $b = b'$  and  $c = c'$ ,  $\varphi(gf(F)v) = g\varphi(f(F)v)$  and  $\varphi(hf(F)v) = h\varphi(f(F)v)$ . It is routine to check that  $\varphi(Ef(F)v) = E\varphi(f(F)v)$  and  $\varphi(Kf(F)v) = K\varphi(f(F)v)$ . So  $\varphi$  is a nonzero homomorphism of  $U_{g,h}$ -modules.  $\square$

For any  $\nu \in \text{Hom}_{\text{alg}}(\mathbb{K}[C, g^{\pm 1}, h^{\pm 1}], \mathbb{K})$ , define a full subcategory  $\mathcal{O}(\nu)$  of  $\mathcal{O}$  as follows:

$$\mathcal{O}(\nu) = \{M \in \mathcal{O} \mid \forall m \in M, z \in \mathbb{K}[C, g^{\pm 1}, h^{\pm 1}], \exists n \in \mathbb{N} \text{ such that } (z - \nu(z))^n m = 0\}.$$

For any  $\nu \in \text{Hom}_{\text{alg}}(\mathbb{K}[C, g^{\pm 1}, h^{\pm 1}], \mathbb{K})$ , suppose  $\nu(C) = \mu$ ,  $\nu(g) = b$ ,  $\nu(h) = c$ . Then  $b, c \in \mathbb{K}^\times$ . Since  $\mathbb{K}$  is an algebraically closed field, there is  $a \in \mathbb{K}$  such that  $\frac{qa + q^{-1}a^{-1}b^2}{(q - q^{-1})^2} = \mu$ . Therefore the Verma module  $V(a, b, c) \in \mathcal{O}(\nu)$  by (4.1), and  $\mathcal{O}(\nu)$  is not empty. By results in [8, Theorem 11.2], we have the following decomposition of  $\mathcal{O}$ .

**Theorem 4.10.** *The category  $\mathcal{O} = \bigoplus_{\nu \in \text{Hom}_{\text{alg}}(\mathbb{K}[C, g^{\pm 1}, h^{\pm 1}], \mathbb{K})} \mathcal{O}(\nu)$ .*

Let  $\mathcal{H}$  be the Harish-Chandra category over  $(U_{g,h}, H)$ , which consists of all  $U_{g,h}$ -modules  $M$  with a simultaneous weight space decomposition for  $H = \mathbb{K}[K^{\pm 1}, g^{\pm 1}, h^{\pm 1}]$ , and finite-dimensional weight spaces. By Proposition 2.5,  $U_{g,h}$  has an anti-involution  $i$ . Thus we can define a duality functor  $F : \mathcal{H} \rightarrow \mathcal{H}$  as follows:  $F(M)$  is the vector space spanned by all  $H$ -weight vectors in  $M^* = \text{Hom}_{\mathbb{K}}(M, \mathbb{K})$ . It is a module under the action determined by

$$\langle am^*, m \rangle = \langle m^*, i(a)m \rangle$$

for  $a \in U_{g,h}$ ,  $m^* \in F(M)$ ,  $m \in M$ . By results in [8],  $F$  defines a duality functor  $F : \mathcal{O} \rightarrow \mathcal{O}^{\text{op}}$ . Moreover,  $F(L(a, b, c)) = L(a, b, c)$ ,  $F(V(a, b, c))$  has the socle  $L(a, b, c)$  and so on.

By Proposition 4.9,  $U_{g,h}$  satisfies the condition (S4) defined in [8]. Therefore it satisfies the conditions (S1), (S2), and (S3) by [8, Proposition 11.3] and [8, Theorem 10.1], where

(S1), (S2) and (S3) are defined in [8]. By [8, Theorem 4.3], we have the following theorem since  $\Gamma$  is trivial.

**Theorem 4.11.** *Let  $\nu \in \text{Hom}_{\mathbb{K}}(\mathbb{K}[K^{\pm 1}, g^{\pm 1}, h^{\pm}], \mathbb{K})$  and  $\mathcal{O}(\nu)$  have the same meaning as in Theorem 4.10. Then:*

- (1) *Each object of the block  $\mathcal{O}(\nu)$  has a filtration whose subquotients are quotients of Verma modules.*
  - (2) *Each block  $\mathcal{O}(\nu)$  has enough projective objects.*
  - (3) *Each block  $\mathcal{O}(\nu)$  is a highest weight category, equivalent to the category of finitely generated right modules over a finite-dimensional  $\mathbb{K}$ -algebra.*
- In particular, BGG Reciprocity holds in  $\mathcal{O}$ .*

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